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To Thomas Stewart Wesner

Dad

To Margot

Phil

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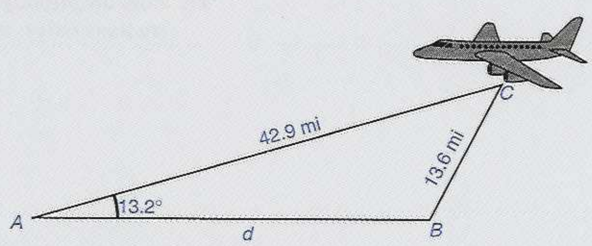
C H A P T E R

8 Additional Topics in Trigonometry

In the next two sections we learn two methods for solving triangles that are not right triangles. After that we introduce vectors, which are used extensively in physics and engineering. We then introduce complex numbers in polar form, which is also very important in physics and engineering. Finally we see another coordinate system besides the usual rectangular coordinate system—the polar coordinate system.

8–1 The law of sines

Ground-based radar at point A determines that the angle of elevation to an aircraft 42.9 miles away is 13.2°. Radar at point B is on a straight line between a point on the ground directly below the aircraft and the radar at A, and determines that the same aircraft is 13.6 miles away from point B. Find the distance from A to B.



A triangle in which none of the angles is a right angle is called an **oblique** triangle. Triangle ABC in the preceding problem is such a triangle. One method for solving certain oblique triangles is the **law of sines.** The following paragraphs develop this law.

First, we observe that *at least two of the angles in every triangle are acute.* If only one angle were acute (less than 90°) then the other two would be obtuse or right (greater than or equal to 90°). This is impossible since the sum of these two angles would be greater than or equal to 180°, and thus the sum of all three angles would be greater than 180°.

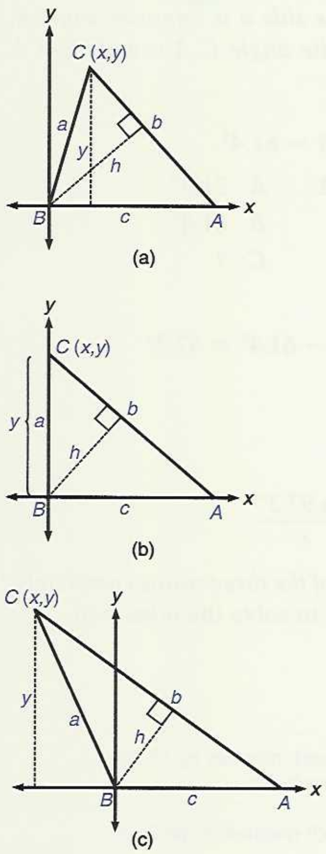


Figure 8–1

Now we consider any triangle ABC, and label two of the acute angles A and C. Angle B may be acute, obtuse, or right. We place the triangle in a coordinate system so that angle B is in standard position and vertex A is on the x-axis. Figure 8–1 shows sketches for the cases where B is (a) acute, (b) right, and (c) obtuse.

From vertex B we construct a line segment perpendicular to side AC and label this line h. From what we know about right triangles we can see that, in all three cases,

$$\sin A = \frac{h}{c} \quad \text{and} \quad \sin C = \frac{h}{a}$$

If we solve for h in each, we obtain

$$h = c \sin A \quad \text{and} \quad h = a \sin C$$

Since $c \sin A$ and $a \sin C$ equal the same quantity (h) they themselves must be equal. Thus,

$$c \sin A = a \sin C$$

$$\frac{c \sin A}{ac} = \frac{a \sin C}{ac} \qquad \text{Divide both members by } ac$$

$$\frac{\sin A}{a} = \frac{\sin C}{c} \qquad \text{Remove common factors}$$

We now derive a similar relation involving angle B and side b. Let (x,y) be the coordinates of the vertex of angle C. From what we know about the trigonometric functions for any angle in standard position (chapter 5) we see, by examining all three cases in the figure, that

$$\sin B = \frac{y}{a} \quad \text{or} \quad y = a \sin B$$

Examining the three triangles, and from what we know about right triangles,

$$\sin A = \frac{y}{b} \quad \text{or} \quad y = b \sin A$$

Thus, $a \sin B = y$ and $y = b \sin A$, so

$$a \sin B = b \sin A$$

$$\frac{\sin A}{a} = \frac{\sin B}{b}$$

Putting these results together we have the law of sines.

The law of sines

In any triangle ABC,

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Concept

The ratio of the sine of an angle to the length of the side opposite that angle is the same for all angles in any triangle.

Note When solving problems only use two of the three ratios at a time.

In the rest of this chapter, we always assume side a is opposite angle A, side b is opposite angle B, and side c is opposite angle C. Example 8–1 A illustrates the law of sines.

■ Example 8–1 A

Solve the triangle ABC if $a = 13.2$, $A = 21.3°$, $B = 61.4°$.

It is a good idea to make a table of values:

a: 13.2	A: 21.3°
b: ?	B: 61.4°
c: ?	C: ?

Find C first.

$$C = 180° - A - B = 180° - 21.3° - 61.4° = 97.3°$$

Now fill in the law of sines.

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

$$\frac{\sin 21.3°}{13.2} = \frac{\sin 61.4°}{b} = \frac{\sin 97.3°}{c}$$

To use the law of sines we must always know one of the three ratios completely. In this case, we know the first ratio, so we use it to solve the other two. Using the first and second ratios:

$$\frac{\sin 21.3°}{13.2} = \frac{\sin 61.4°}{b}$$

$$b \sin 21.3° = 13.2 \sin 61.4°$$ Multiply each member by $13.2b$ (or cross multiply)

$$b = \frac{13.2 \sin 61.4°}{\sin 21.3°} \approx 31.9$$ Divide each member by sin 21.3°

Using the first and third ratios:

$$\frac{\sin 21.3°}{13.2} = \frac{\sin 97.3°}{c}$$

$$c = \frac{13.2 \sin 97.3°}{\sin 21.3°} \approx 36.0$$

Thus we have solved the triangle:

a: 13.2	A: 21.3°
b: 31.9	B: 61.4°
c: 36.0	C: 97.3°

■

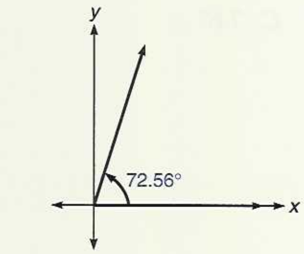
The ambiguous case

If we are given only one of the two angles of a triangle it is possible to get two different solutions to the problem. The reason for this is shown in example 8–1B. When we use the law of sines to solve a triangle *for which only one angle is known* we call this the **ambiguous case.**

■ ***Example 8–1 B***

Solve triangle ABC if $a = 28.5$, $b = 30.0$, $A = 65°$.

a: 28.5	A: 65°	Make a table of values
b: 30.0	B: ?	
c: ?	C: ?	



$$\frac{\sin 65°}{28.5} = \frac{\sin B}{30.0} = \frac{\sin C}{c}$$ Fill values into the law of sines

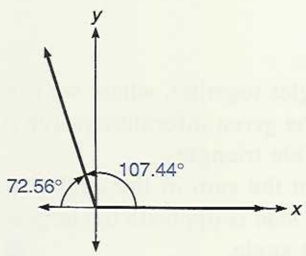
$$\frac{\sin 65°}{28.5} = \frac{\sin B}{30.0}$$ Use the first two ratios to find angle B

$$30.0 \sin 65° = 28.5 \sin B$$

$$\frac{30.0 \sin 65°}{28.5} = \sin B$$

A *reference angle* for angle B is found with the inverse sine function.

$$B' = \sin^{-1}\left(\frac{30.0 \sin 65°}{28.5}\right) \approx 72.56°$$



Since angle B is in a triangle, we know its measure is between 0° and 180°. Thus, using B' as a reference, angle B could either be 72.56° or 180° − 72.56° = 107.44°. See the figure.

At this point we must divide the problem into two cases: the case where $B \approx 72.56°$ and the one where $B \approx 107.44°$.

Case 1: $B \approx 72.56°$

a: 28.5	A: 65°
b: 30.0	B: 72.56°
c: ?	C: ?

$$C \approx 180° - 65° - 72.56° \approx 42.44°$$

We can use the value of angle C to find c.

$$\frac{\sin 65°}{28.5} \approx \frac{\sin 42.44°}{c}$$

$$c \approx \frac{28.5 \sin 42.44°}{\sin 65°} \approx 21.2$$

Thus, $C \approx 42.4°$, $c \approx 21.2$.

Case 2: $B \approx 107.44°$

a: 28.5	A: 65°
b: 30.0	B: 107.44°
c: ?	C: ?

$$C \approx 180° - 65° - 107.44° \approx 7.56°$$

$$\frac{\sin 65°}{28.5} \approx \frac{\sin 7.56°}{c}$$

$$c \approx \frac{28.5 \sin 7.56°}{\sin 65°} \approx 4.1$$

Thus, $C \approx 7.6°$, $c \approx 4.1$.

We can summarize these two solutions in two tables.

Case 1		Case 2	
a: 28.5	A: 65°	a: 28.5	A: 65°
b: 30.0	B: 72.6°	b: 30.0	B: 107.4°
c: 21.2	C: 42.4°	c: 4.1	C: 7.6°

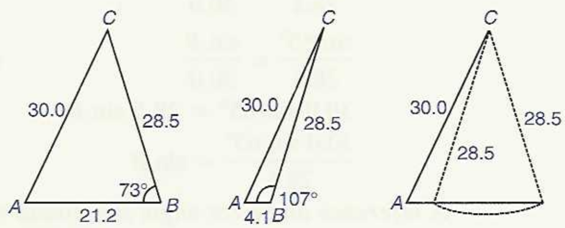


Figure 8–2

Figure 8–2 shows the two triangles and both triangles together, where we can see why the ambiguous case was possible—with the given information side a could be in one of two positions, giving two possible triangles.

As a final check on our work, we observe that the sum of the angles in each case is 180°, and that in each case the longest side is opposite the largest angle and the shortest side is opposite the smallest angle.

The ambiguous case does not always produce two triangles. When only one triangle is possible, the measure of the third angle will be a negative number in case 2.

Mastery points

Can you

- State and use the law of sines to solve oblique triangles?
- Recognize and solve the ambiguous case when using the law of sines?

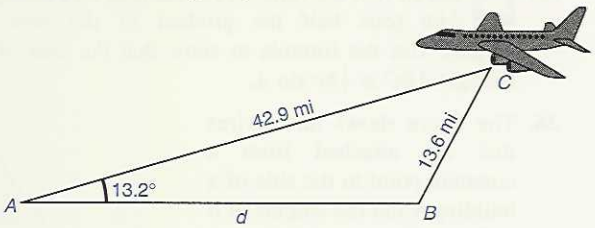
Exercise 8–1

In the following problems, round answers to the same number of decimal places as the data, unless otherwise specified.

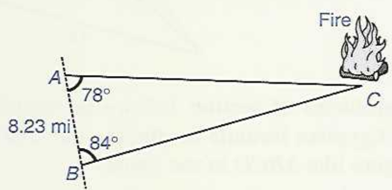
Solve the following oblique triangles using the law of sines.

1. $a = 12.5, A = 35°, B = 49°$
2. $b = 17.1, B = 100°, C = 10°$
3. $a = 1.25, B = 13.6°, C = 132°$
4. $c = 9.04, A = 51.6°, B = 40.0°$
5. $b = 92.5, A = 47°, B = 100°$
6. $c = 10.2, A = 16.7°, B = 89.2°$
7. $a = 0.452, A = 67.6°, C = 91.8°$
8. $b = 0.508, B = 13.1°, C = 5.2°$
9. $c = 5.00, A = 100°, B = 45°$
10. $a = 10.9, B = 76.9°, C = 100°$
11. $a = 12.5, b = 13.2, B = 49°$
12. $b = 37.1, c = 21.3, B = 100°$
13. $a = 4.25, c = 2.86, A = 132°$
14. $c = 9.04, a = 21.3, C = 10.0°$
15. $b = 92.5, c = 98.6, B = 43.7°$
16. $c = 10.2, a = 16.7, A = 89.2°$
17. $a = 4, b = 22, A = 30°$
18. $a = 0.452, c = 0.606, C = 91.8°$
19. $b = 6.35, c = 4.29, C = 42.3°$
20. $b = 0.508, c = 1.09, C = 5.2°$
21. $c = 5.00, b = 8.00, B = 45.0°$
22. $a = 10.9, c = 16.9, C = 100.0°$

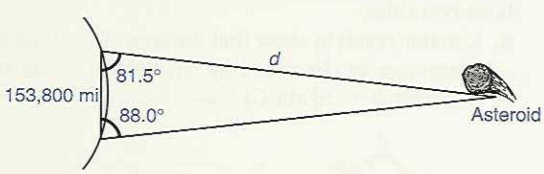
23. Ground-based radar at point A determines that the angle of elevation to an aircraft 42.9 miles away is 13.2°. Radar at point B is on a straight line between a point on the ground directly below the aircraft and the radar at A and determines that the same aircraft is 13.6 miles away from point B. To the nearest 0.1 mile, find the distance from A to B (see the diagram). Assume $B > 90°$.



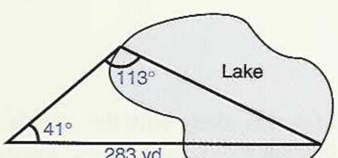
24. Two forest rangers sight a fire. Their reports are plotted on a map and yield the results shown in the diagram. If the locations of the rangers are 8.23 miles apart, how far is the fire from station A, to the nearest 0.1 mile?



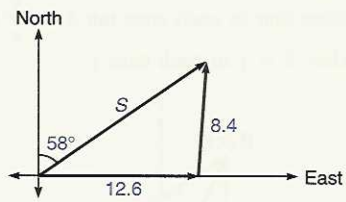
25. The diagram shows a situation in which astronomers made measurements at two locations of a slow-moving asteroid. Using their measurements, find the distance d to the asteroid, to the nearest 100 miles.



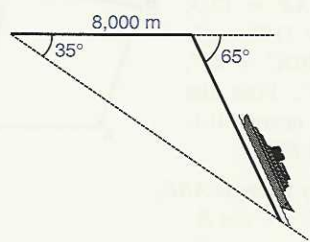
26. A surveyor made the measurements shown in the diagram. Find the distance across the lake, to the nearest foot.



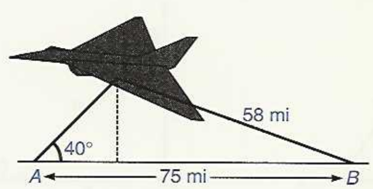
27. The diagram illustrates a situation in which a ship is moving at 12.6 knots heading due east. It is moving through a current moving east of north at 8.4 knots. The result is that the ship is moving at an angle of 58° east of north. Find the true speed S of the ship.



28. A ship travels due east to a point 8,000 meters from its starting point. It then turns toward the south through an angle of 65° and proceeds until it crosses a line of sight from the starting position to itself that is 35° south of east. At this point, how far is the ship from its starting point, to the nearest 10 meters?

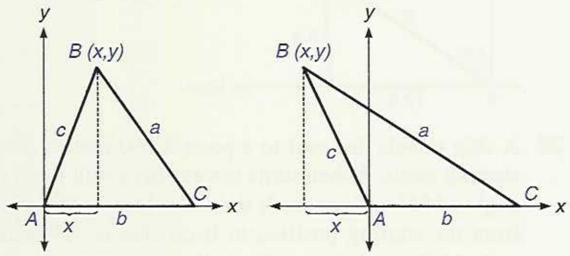


29. Two cities are 75 miles apart. An aircraft that is between the two cities is being tracked from radar in each city. City A's radar shows that the aircraft is at an angle of elevation of 40°; city B's radar shows that the slant distance of the plane to city B is 58 miles and that the angle of elevation there is less than 40°. What is the slant distance d from the plane to city A, to the nearest mile?

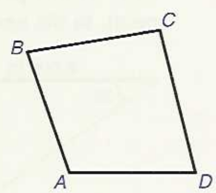


30. Find the height of the aircraft in problem 29, to the nearest 100 feet (1 mile = 5,280 feet). Use the diagram for help.

31. Show that in any oblique triangle *ABC*, if h is the altitude of the triangle relative to side b, then an expression for h is $h = \dfrac{b \cdot \tan A \cdot \tan C}{\tan A + \tan C}$. (*Hint:* The diagram shows a triangle in which angle A is acute (on the left) and obtuse. Note that in each case $\tan A = \dfrac{y}{x}$ and $\tan C = \dfrac{y}{b - x}$. Also, $h = y$ in each case.)



32. Quadrilateral *ABCD* is shown in the diagram, where $AB = 17.3$, $AD = 18.9$, angle $A = 110°$, angle $ABD = 52°$, angle $BDC = 41°$, and angle $C = 93°$. Find the length of *CD* to the nearest 0.1. (*Hint:* Draw diagonal *BD*.)



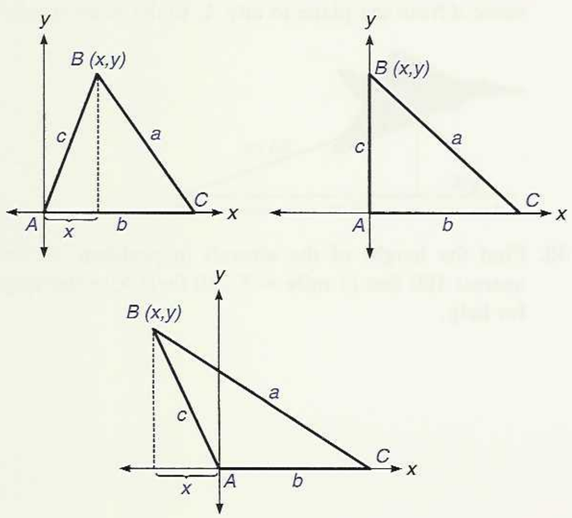
33. Show that in any triangle *ABC*,

[1] $a = b \cos C + c \cos B$

[2] $b = c \cos A + a \cos C$

[3] $c = a \cos B + b \cos A$

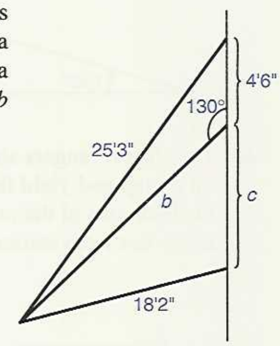
(*Hint:* The diagram shows angle A in standard position when the angle is acute, right, or obtuse. Show why the statements $\cos A = \dfrac{x}{c}$ and $\cos C = \dfrac{b - a}{x}$ are true in each case, and then use them to show that [2] is true.)



34. An army observation point is 325 yards northeast (i.e., 45° north of east) of a second point. At this point, a tank is sighted on a line of sight 37° south of east. The same tank is sighted at the second point along a sight 18° north of east. To the nearest yard, how far is the tank from the first observation point?

35. Recall that the formula for the area of a triangle is $\frac{1}{2}bh$ (one half the product of the base and height). Use the formula to show that the area of any triangle *ABC* is $\frac{1}{2}bc \sin A$.

36. The figure shows three wires that are attached from a common point to the side of a building. Find the lengths of b and c, to the nearest inch.

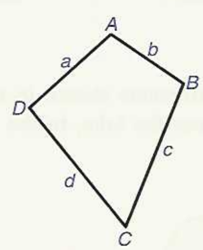


37. In the problems of section 1–2 we presented the ancient Egyptian formula for finding the area of a four-sided figure like *ABCD* in the figure:

$$\tfrac{1}{4}(ab + ad + bc + cd)$$

and noted that it is inaccurate. Problem 35 shows that the area of any triangle *ABC* is $\frac{1}{2}bc \sin A$. It is also $\frac{1}{2}ac \sin B$ and $\frac{1}{2}ab \sin C$. Geometrically, this is one half the product of two sides and the sine of the angle between those two sides.

a. Use this result to show that the area of the four-sided figure can be described by $\frac{1}{4}(ab \sin A + ad \sin D + bc \sin B + cd \sin C)$.



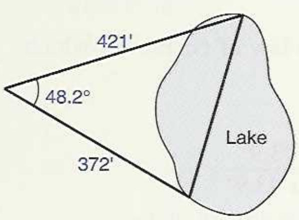
b. Use the result of part a, along with the fact that for $0 < \theta < 180°$, $0 < \sin \theta < 1$, to show that the Eygptian formula is always too large, except for rectangles, when it is exact. (Assume angles A, B, C, and D are all less than 180° in measure.)

Skill and review

1. Solve the right triangle ABC if $a = 6.5$ and $c = 9.0$. Round answers to the nearest 0.1.
2. Solve the trigonometric equation $\sec x = -\frac{2}{3}\sqrt{3}$ for all primary solutions, in radians.
3. Verify the identity $(\sin\theta - \sec\theta)(\csc\theta + \cos\theta) = \dfrac{\sin^2\theta\cos^2\theta - 1}{\sin\theta\cos\theta}$.
4. Factor $2x^5 - x^4 - 10x^3 + 5x^2 + 8x - 4$.
5. What is the length of the arc determined by a central angle of 24° on a circle with radius 18 meters?

8–2 The law of cosines

A surveyor made the measurements shown in the diagram to calculate the distance across a lake. Compute this distance.



The triangle in this problem is an oblique triangle. Unfortunately it cannot be solved by the law of sines. This is because we do not know any of the three ratios completely. In these cases we use the law of cosines, which we develop in this section. We use the distance formula to develop this law.

Let triangle ABC be any triangle. Put angle C in standard position and side a on the positive portion of the x-axis. See figure 8–3. Let (x,y) be the point at vertex A. (The figure shows C as an obtuse angle, but the algebraic statements that follow would also apply if C were acute or right.) Recall the distance formula (section 3–1):

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

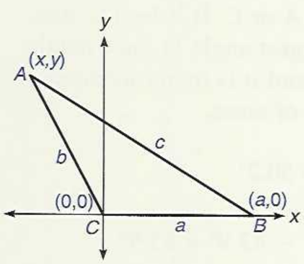


Figure 8–3

Apply this to length c. Use $(x_2,y_2) = (a,0)$, and $(x_1,y_1) = (x,y)$.

$$c = \sqrt{(a - x)^2 + (0 - y)^2}$$
$$c^2 = (a - x)^2 + y^2$$
$$c^2 = a^2 - 2ax + x^2 + y^2$$

We know that $x^2 + y^2 = b^2$ (also by the distance formula).

$$c^2 = a^2 - 2ax + b^2 \qquad \text{Replace } x^2 + y^2 \text{ by } b^2$$
$$c^2 = a^2 + b^2 - 2ax$$

We know that $\cos C = \dfrac{x}{b}$ by the definition of the cosine function (section 5–3). Thus, $x = b\cos C$.

$$c^2 = a^2 + b^2 - 2ab\cos C \qquad \text{Replace } x \text{ by } b\cos C$$

This result is called the **law of cosines** for angle C. If we put angle A or angle B in standard position, we arrive at two other versions of this law.

The law of cosines

For any triangle ABC,

$$a^2 = b^2 + c^2 - 2bc \cos A$$
$$b^2 = a^2 + c^2 - 2ac \cos B$$
$$c^2 = a^2 + b^2 - 2ab \cos C$$

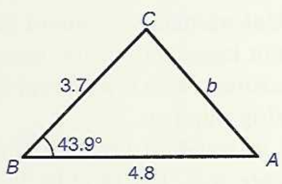
There is a pattern in each of the three equations above. The variable in the left member is the length of the side opposite the angle in the right member. The sides in the right member appear twice.

The law of cosines should be used whenever the law of sines cannot be used. In particular, the law of cosines is used when we know the lengths of two sides and the angle included between those sides, or when we know the lengths of all three sides. Example 8–2 A illustrates both situations.

■ *Example 8–2 A*

Solve the triangle.

1. $a = 3.7$, $c = 4.8$, $B = 43.9°$



Since we know angle B we use the form of the law of cosines in which angle B appears.

$$b^2 = a^2 + c^2 - 2ac \cos B$$
$$b^2 = 3.7^2 + 4.8^2 - 2(3.7)(4.8) \cos 43.9°$$
$$b = \sqrt{3.7^2 + 4.8^2 - 2(3.7)(4.8) \cos 43.9°}$$
$$b \approx 3.337 \quad \text{We will round this to 3.3 in the final answer}$$

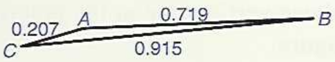
We can now use the law of sines to find angle A or C. It is best to find angle A first, because we know it is not the largest angle in the triangle (angle C is because side c is the longest side) and it is therefore acute. This eliminates the ambiguous case of the law of sines.

$$\frac{3.7}{\sin A} = \frac{3.337}{\sin 43.9°}, \text{ so } A \approx 50.2°$$

$$C = 180° - A - B \approx 180° - 50.2° - 43.9° \approx 85.9°$$

$a = 3.7$ $\quad A \approx 50.2°$
$b \approx 3.3$ $\quad B = 43.9°$
$c = 4.8$ $\quad C \approx 85.9°$

Note It is never necessary to use the law of cosines more than once to solve a triangle. Complete the solution with the law of sines.



2. $a = 0.915$, $b = 0.207$, $c = 0.719$

We do not know any of the angles, so we cannot use the law of sines. (Without any angles we cannot know the value of any of the three ratios in the law of sines.) Thus, we use the law of cosines to find one of the angles.

It is best to find the largest angle first (see the following note); this will be angle A since it is opposite the longest side a. We must use the form of the law of cosines that includes angle A.

$$a^2 = b^2 + c^2 - 2bc \cos A$$

We solve for $\cos A$ before replacing the variables with known values.

$$2bc \cos A = b^2 + c^2 - a^2$$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

Substituting known values, we get

$$\cos A = \frac{0.207^2 + 0.719^2 - 0.915^2}{2(0.207)(0.719)} \approx -0.9320$$

Since $\cos A < 0$ we know A is in quadrant II (obtuse). The inverse cosine function gives results in quadrants I and II, so we use the inverse cosine function directly.

$$A \approx \cos^{-1}\left(\frac{0.207^2 + 0.719^2 - 0.915^2}{2(0.207)(0.719)}\right) \approx 158.74°$$

Now that we know the measure of the largest angle in the triangle, we can use the law of sines or the law of cosines to find another angle. The law of sines is generally easier to calculate. Using the law of sines we find one of the acute angles: $B \approx 4.70°$.

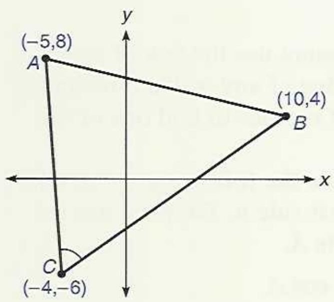
$$C = 180° - 158.74° - 4.70° \approx 16.56°$$

The triangle is solved since we now know the measures of the three sides and three angles.

$a = 0.915$	$A \approx 158.7°$
$b = 0.207$	$B \approx 4.7°$
$c = 0.719$	$C \approx 16.6°$

Note 1. There is no ambiguous case when using the law of cosines because the range of the inverse cosine function is $0° \le y \le 180°$, which includes the range of the measure of the angles in a triangle, $0° < y < 180°$.

2. When solving a triangle remember that only one angle (the largest) can be obtuse (measure more than 90°), and the largest angle is opposite the longest side.



3. Find the measure of angle C in a triangle whose vertices are at the points $A(-5,8)$, $B(10,4)$, and $C(-4,-6)$. See the figure.

We use the distance formula to find the lengths AB, BC, AC.

$$AB = \sqrt{(10-(-5))^2 + (4-8)^2} = \sqrt{241}$$
$$BC = \sqrt{(-4-10)^2 + (-6-4)^2} = 2\sqrt{74}$$
$$AC = \sqrt{(-4-(-5))^2 + (-6-8)^2} = \sqrt{197}$$

To find angle C, we use the version of the law of cosines with angle C.

$$c^2 = a^2 + b^2 - 2ab\cos C$$
$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$
$$= \frac{(2\sqrt{74})^2 + (\sqrt{197})^2 - (\sqrt{241})^2}{2(2\sqrt{74})(\sqrt{197})}$$
$$= \frac{296 + 197 - 241}{4(\sqrt{74})(\sqrt{197})} \approx 0.5218$$
$$C \approx 58.5^\circ$$

■

Mastery points

Can you

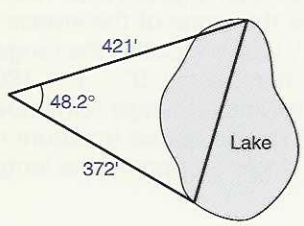
- Use the law of cosines to solve triangles when the law of sines cannot be used?

Exercise 8–2

Solve the following oblique triangles. You will have to use the law of cosines as the first step. Round answers to the same number of decimal places as the data.

1. $a = 3.2$, $b = 5.9$, $C = 39.4^\circ$
2. $a = 4.9$, $b = 3.2$, $C = 78.2^\circ$
3. $b = 61.3$, $c = 23.9$, $A = 124.0^\circ$
4. $b = 123.0$, $c = 89.4$, $A = 19.5^\circ$
5. $a = 31.4$, $c = 17.0$, $B = 100.3^\circ$
6. $a = 67.25$, $c = 13.56$, $B = 76.30^\circ$
7. $a = 23.5$, $b = 19.4$, $c = 35.0$
8. $a = 61.7$, $b = 80.0$, $c = 102.0$
9. $a = 0.214$, $b = 0.500$, $c = 0.399$
10. $a = 1.75$, $b = 0.98$, $c = 1.03$
11. $a = 13.2$, $b = 5.9$, $C = 139.4^\circ$
12. $a = 14.9$, $b = 13.2$, $C = 45.0^\circ$
13. $b = 61.3$, $c = 43.9$, $A = 24.5^\circ$
14. $b = 23.9$, $c = 89.4$, $A = 79.5^\circ$
15. $a = 30.0$, $c = 20.0$, $B = 112.0^\circ$
16. $a = 6.72$, $c = 1.55$, $B = 76.35^\circ$
17. $a = 235$, $b = 194$, $c = 354$
18. $a = 61.7$, $b = 80.0$, $c = 42.0$
19. $a = 0.21$, $b = 0.49$, $C = 1.50^\circ$
20. $a = 10.0$, $b = 13.9$, $c = 17.5$

21. A surveyor made the measurements shown in the diagram to calculate the distance across a lake. Compute the distance to the nearest 0.1 foot.



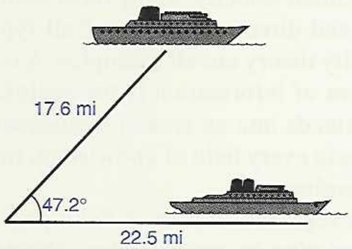
22. Calculate the measure of the smallest angle in a triangle whose sides have measure 22.1 cm, 32.6 cm, and 40.5 cm.

23. Calculate the measure of the largest angle in a triangle whose sides have measure 12.3 in., 16.2 in., and 19.0 in.

24. A numerically controlled laser cloth cutter is being set up to cut a triangular pattern. The vertices of the triangle are at $A(2,6)$, $B(5,8)$, and $C(6,2)$. (a) Find the length of side AB. (b) Find the measure of the smallest angle in the triangle.

25. In the same situation as problem 24 the vertices of another triangular piece of cloth are determined to be at $A(0,5)$, $B(2,3)$, and $C(8,4)$. Determine the measure of the largest of the three angles, A, B, or C, to the nearest 0.1°.

26. Two ships are being tracked by radar. One ship is determined to be at 17.6 miles from the radar, while the second is 22.5 miles from the radar. The lines of sight from the radar to the two ships form an angle of 47.2° (see the diagram). Find the distance between the two ships to the nearest 0.1 mile.



27. In the situation described in problem 26, what would be the angle formed by the two lines of sight to the ships if the ships were 31.5 miles apart?

28. A ship leaves a harbor heading due east and travels 17.3 kilometers (km). It then turns north through a 33° angle and travels for another 22.0 km. How far is the ship from its starting point, to the nearest kilometer?

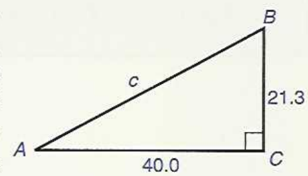
29. A plane takes off and travels southeast (45° south of east) for 27 miles, then turns due south and travels for 16 miles. How far is it from its starting position, to the nearest mile?

30. The points $(5,3)$, $(-2,1)$, and $(1,-4)$ form a triangle. Find the measure of the smallest angle in this triangle, to the nearest 0.1°.

31. Find the measure of the largest angle in the triangle of problem 30, to the nearest 0.1°.

32. Two observers are 87 meters apart, and a rangefinder shows that a certain building is 111 meters from one observer and 114 meters from the other. What is the angle formed by the two lines of sight from the building to the observers, to the nearest degree?

33. The triangle in the diagram is a right triangle. Can the law of cosines be used to find the length of c? Compare using the law of cosines to using the Pythagorean theorem.

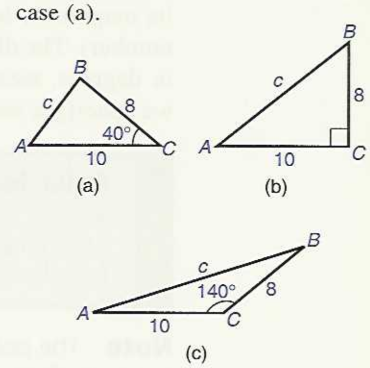


34. The diagram shows three triangles in which angle C is (a) acute, (b) right, and (c) obtuse. If we use the Pythagorean theorem and apply it to the right triangle in (b) and apply the law of cosines to angle C in all three situations, the resulting equations are the two equations

$$c^2 = a^2 + b^2$$
$$c^2 = a^2 + b^2 - 2ab \cos C$$

We can view the term $-2ab \cos C$ as a "correction factor" to the Pythagorean theorem that will give the correct result even when angle C is not 90°.

Solve for side c in cases (a), (b), and (c), using the law of cosines. Discuss how the correction factor increases the length of side c, compared to case (b), in case (c), and how it decreases it compared to case (b), as in case (a).



Skill and review

1. Solve the triangle ABC if $a = 6.8$, $b = 12.0$, and $B = 43°$.
2. Solve the right triangle ABC if $a = 6.8$ and $B = 43°$.
3. Graph the function $f(x) = 2x^4 + 5x^3 - 8x^2 - 17x - 6$.
4. Use the graph of $y = x^2$ to graph the function $f(x) = -(x - 3)^2 + 1$.
5. If $\sin \theta = -\frac{2}{3}$ and $\cos \theta < 0$, find the value of $\tan \theta$.

8–3 Vectors

A sled loaded with lumber will not move until the horizontal component of the applied force is 1,200 pounds or more. If a winch being used to move the sled can apply a maximum force of 1,700 pounds, what is the largest angle of elevation at which the winch can act on the sled and move it?

This problem is one application of vectors, which we study in this section.

If we know that a plane flying over a certain spot is flying at 100 mph, we cannot tell where it will be in 1 hour without also knowing its direction. This combination of speed and direction is called *velocity*. Many other natural phenomena are described by a magnitude and direction: forces of all types, accelerations, alternating voltage in electricity theory are all examples. A conceptual tool used to describe two such pieces of information is the **vector.** It is noteworthy that the concept of vectors extends into an area of mathematics called *linear algebra,* which has applications in every field of knowledge, from physics and economics to medicine and sociology.

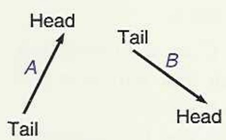


Figure 8–4

We imagine a vector as a directed line segment. That is, a finite portion of a straight line that is considered to be pointing in one direction. Our representation of vectors would commonly be called "arrows." Examples of vectors are shown in figure 8–4. Observe that we use the terms **head** and **tail** to describe the "end" and "beginning" of a vector, respectively, and that we often use capital letters, such as A and B, to name a vector.

One way to describe a vector is by specifying its length and direction. The length is called the **magnitude** of the vector, and for a vector A we denote its magnitude by $|A|$ (the same notation as that for absolute value of a real number). The **direction** of a vector is specified by an angle, θ, usually specified in degrees, measured in the same way as angles in standard position. When we specify a vector this way we say it is in **polar form.**

Polar form of a vector

A vector A in polar form is the ordered pair $A = (|A|,\theta_A)$.
$|A|$ is the magnitude of vector A; $|A| \geq 0$.
θ_A is the direction of vector A.

Note The polar form of a vector is not unique; all coterminal values of θ_A are equivalent.

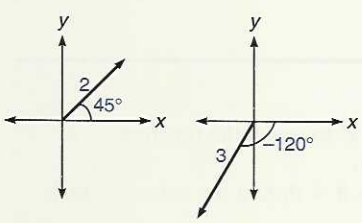


Figure 8–5

Figure 8–5 illustrates the vectors (2,45°) and (3,−120°).

A vector can also be specified by using its *relative rectangular coordinates.* For a vector in standard position, this is equivalent to specifying the coordinates of its head (see figure 8–7). If a vector is not in standard position, its relative rectangular coordinates are relative to the coordinates of the vector's tail. We refer to these coordinates as describing the **rectangular form** of a vector.

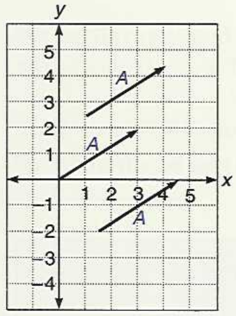


Figure 8–6

Rectangular form of a vector
A vector A in rectangular form is $A = (A_x, A_y)$, where A_x is called the **horizontal component** of the vector and A_y is called the **vertical component** of the vector.

Figure 8–6 illustrates the vector $A(3,2)$, shown in three different positions.

Converting polar form to rectangular form

There is a useful relation for converting the polar form of a vector to its rectangular form. If we recall the definitions of section 5–2 for an angle in standard position, we can see the following.

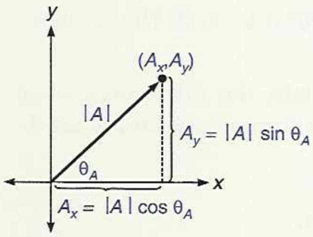


Figure 8–7

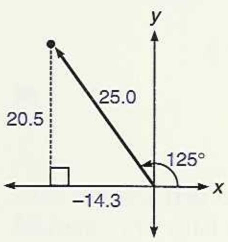
To convert polar form to rectangular form
Given vector $A = (|A|,\theta_A) = (A_x, A_y)$,

$$A_x = |A| \cos \theta_A \quad \text{and} \quad A_y = |A| \sin \theta_A$$

This is true because, by the definitions of section 5–2, $\cos \theta_A = \dfrac{A_x}{|A|}$ and $\sin \theta_A = \dfrac{A_y}{|A|}$. Figure 8–7 illustrates this relationship, where we view A_x and A_y as directed horizontal and vertical distances, respectively.

Example 8–3 A illustrates converting a vector in polar to form to its rectangular form.

■ ***Example 8–3 A***



Convert the polar form of the vector to the rectangular form (approximate to the nearest tenth). $A = (25.0,125°)$.

$$A_x = |A| \cos \theta_A = 25.0 \cos 125° \approx -14.3$$
$$A_y = |A| \sin \theta_A = 25.0 \sin 125° \approx 20.5$$

Thus, the rectangular form is $(-14.3,20.5)$. ■

Converting rectangular form to polar form

When we convert from rectangular to polar form we always give the direction of the vector θ_V so that it has the smallest possible absolute value. This means *we will choose* θ_V *so that* $-180° < \theta_V \leq 180°$. One reason for doing this is that this is the result obtained from electronic calculators (see the discussion following example 8–3 B).

Examining figure 8–7 shows that, for a given vector $V = (V_x,V_y) = (|V|,\theta_V)$, $|V| = \sqrt{V_x^2 + V_y^2}$, and $\tan \theta_V' = \dfrac{V_y}{V_x}$ if $V_x \neq 0$. The angle $\theta_V' = \tan^{-1}\dfrac{V_y}{V_x}$ is only the reference angle and θ_V depends on the quadrant in which the vector occurs.

Reference angles obtained with the inverse tangent function fall in the range $-90° < \theta' < 90°$ (quadrant I and quadrant IV). If $V_x > 0$ this is the value of θ_V, since that angle should be in quadrant I or IV.

If $V_x < 0$, θ_V can be obtained by adding or subtracting 180° to or from θ'_V. If $\theta'_V < 0$, add 180°, and if $\theta'_V > 0$, subtract 180°. This can be summarized as follows.

To convert rectangular form to polar form

Given vector $V = (V_x, V_y) = (|V|, \theta_V)$. Then,

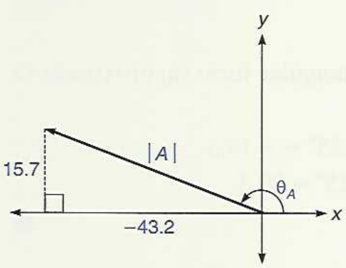
$$|V| = \sqrt{V_x^2 + V_y^2},\ \theta'_V = \tan^{-1}\frac{V_y}{V_x} \text{ if } V_x \neq 0, \text{ and}$$

$$\theta_V = \begin{cases} \theta'_V & \text{if } V_x > 0 \\ \theta'_V - 180° \text{ if } \theta'_V > 0 & \text{if } V_x < 0 \\ \theta'_V + 180° \text{ if } \theta'_V < 0 & \end{cases}$$

Note If $V_x = 0$ then θ_V is 90° if $V_Y > 0$, and $-90°$ if $V_Y < 0$. This is clear from a sketch of the vector.

The examples and the exercises will make clear why the rule works when $V_x < 0$. Example 8–3 B illustrates finding the polar form of a vector when its rectangular form is known.

■ *Example 8–3 B*



Vector $A = (-43.2, 15.7)$. Find the polar form of A.

$$|A| = \sqrt{A_x^2 + A_y^2}$$
$$= \sqrt{(-43.2)^2 + (-15.7)^2} \approx 46.0 \quad \text{Nearest tenth}$$
$$\theta'_A = \tan^{-1}\left(\frac{15.7}{-43.2}\right) \approx -20.0°$$
$$\theta_A \approx -20° + 180° \approx 160° \quad A_x < 0, \theta'_V < 0$$

Thus, $A \approx (46.0, 160°)$. ■

Using special calculator keys

Most engineering/scientific calculators are programmed to perform the conversions of examples 8–3 A and 8–3 B. These calculators have keys marked ''R→P'' or simply ''→P'' (rectangular to polar conversion) and ''P→R'' or ''→R'' (polar to rectangular conversion), or something equivalent. The results are stored in locations referred to as x and y. Typical keystrokes are illustrated here. Example 8–3 A would be done as follows. (The TI-81 is discussed below.)

$A = (25.0, 125°)$

25 $\boxed{\text{P} \to \text{R}}$ 125 $\boxed{=}$ Display: $\boxed{-14.33941091}$

$\boxed{x \leftrightarrow y}$ Display: $\boxed{20.47880111}$

Thus, $A \approx (-14.3, 20.5)$.

Example 8–3 B would be done in the following way.

$A = (-43.2, 15.7)$

43.2 $\boxed{+/-}$ $\boxed{R\rightarrow P}$ 15.7 $\boxed{=}$ Display: $\boxed{45.96444278}$

$\boxed{x\leftrightarrow y}$ Display: $\boxed{160.0275433}$

Thus, $A \approx (46.0, 160°)$.

The TI-81 uses the values X, Y, R, θ. (Y, R, θ are $\boxed{\text{ALPHA}}$ 1, $\boxed{\times}$, and 3, respectively.) It also uses the two MATH functions "R ▸ P(" (Rectangular to polar) and "P ▸ R(" (Polar to rectangular).

Example 8–3 A:

$\boxed{\text{MODE}}$ Deg $\boxed{\text{ENTER}}$ Make sure the calculator is in degree mode.

$\boxed{\text{MATH}}$ 2 25 $\boxed{\text{ALPHA}}$ $\boxed{.}$ 125 $\boxed{)}$ $\boxed{\text{ENTER}}$

Display: $\boxed{-14.33941091}$

$\boxed{\text{ALPHA}}$ 1 $\boxed{\text{ENTER}}$ Display: $\boxed{20.47880111}$

Example 8–3 B:

$\boxed{\text{MATH}}$ 1 $\boxed{(-)}$ 43.2 $\boxed{\text{ALPHA}}$ $\boxed{\cdot}$ 15.7 $\boxed{)}$ $\boxed{\text{ENTER}}$

Display: $\boxed{45.96444278}$

$\boxed{\text{ALPHA}}$ 3 $\boxed{\text{ENTER}}$ Display: $\boxed{160.0275433}$

Addition of vectors

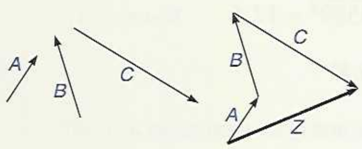


Figure 8–8

It has been shown experimentally that natural phenomena that are described by vectors combine as if they were connected tail to head in a series. The result is a vector with its tail at the tail of the first vector in the series, and its head at the head of the last vector in the series. The resulting vector is called the **resultant vector.** This is illustrated in figure 8–8, where vectors *A, B,* and *C* combine into the resultant vector *Z.* This idea is the basis for the definition of addition of vectors, which we develop here.

Example 8–3 C illustrates that this process is equivalent to summing all the horizontal components and, separately, the vertical components. This is the basis for the definition of the addition of two vectors.

Vector sum

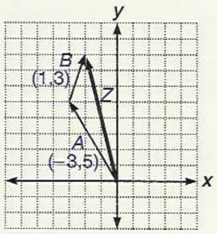
Let Z be the resultant (vector sum) of two vectors $A = (A_x, A_y)$ and $B = (B_x, B_y)$. Then we say $Z = A + B$, where

$$Z_x = A_x + B_x \quad \text{and} \quad Z_y = A_y + B_y$$

Observe that this definition describes how to add two vectors whose rectangular form is known. When vectors are known in polar form they must first be converted to rectangular form. This is illustrated in example 8–3 C.

■ *Example 8–3 C*

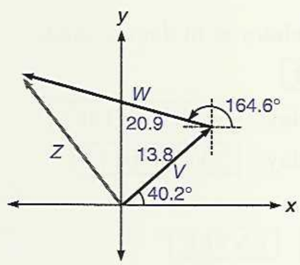
Find the vector sum of the given vectors.



1. $A = (-3,5)$ and $B = (1,3)$

$$Z_x = A_x + B_x = -3 + 1 = -2$$
$$Z_y = A_y + B_y = 5 + 3 = 8$$
$$Z = (-2,8)$$

This is illustrated in the figure.

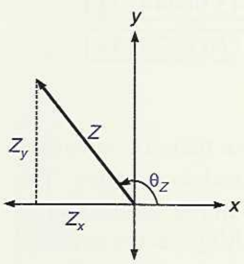


2. $V = (13.8,40.2°)$, $W = (20.9,164.6°)$

Note that these vectors are in polar form. See the figure.

$$\begin{aligned} Z_x &= V_x + W_x \\ &= |V| \cos \theta_V + |W| \cos \theta_W \\ &= 13.8 \cos 40.2° + 20.9 \cos 164.6° \approx -9.6092 \\ Z_y &= V_y + W_y \\ &= |V| \sin \theta_V + |W| \sin \theta_W \\ &= 13.8 \sin 40.2° + 20.9 \sin 164.6° \approx 14.4574 \end{aligned}$$

Thus, $Z \approx (-9.6,14.5)$ (to nearest tenth).



We should find Z in polar form, since V and W were given in this form.

$$Z = \sqrt{Z_x^2 + Z_y^2} = \sqrt{(-9.6093)^2 + 14.4589^2} \approx 17.4$$ Nearest 0.1

$$\theta'_Z = \tan^{-1}\frac{Z_y}{Z_x} \approx \tan^{-1}\left(\frac{14.4589}{-9.6093}\right) \approx -56.4°$$

$$\theta_Z = \theta'_Z + 180° \approx 123.6°$$ Since $Z_x < 0$, and θ'_Z is negative, we add 180°

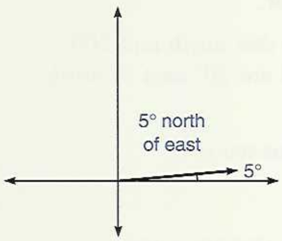
Thus, in polar form, $Z \approx (17.4,123.6°)$. ■

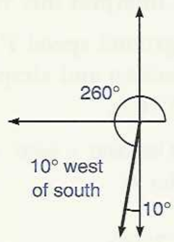
Problem 76 in the exercises presents a program that adds two or more vectors given in polar form.

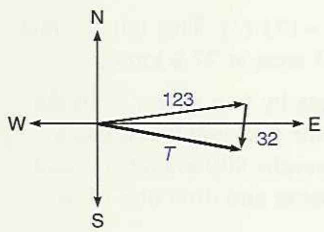
The next example illustrates a general principle of navigating aircraft (and, analogously, ships at sea). The speed of the aircraft relative to the air is called the *airspeed*, and the direction in which the aircraft is pointed is its *heading*. The airspeed and heading combine into the *heading vector H*. The *wind vector W* is the speed and direction of the wind. If we add these two vectors we get the *true course* and *ground speed* of the plane, $T = H + W$, where T is the speed and direction relative to the earth's surface.

■ ***Example 8–3 D***

An aircraft is flying with heading 5° north of east and airspeed 123 knots. The wind is blowing at 32 knots in the direction 10° west of south. Find the true course and ground speed of the aircraft (see the figure).







$$H = (123,5°);\ W = (32,260°)$$
$$T = H + W$$
$$\begin{aligned} T_x &= H_x + W_x \\ &= |H|\cos\theta_H + |W|\cos\theta_W \\ &= 123\cos 5° + 32\cos 260° \approx 116.98 \\ T_y &= H_y + W_y \\ &= |H|\sin\theta_H + |W|\sin\theta_W \\ &= 123\sin 5° + 32\sin 260° \approx -20.79 \end{aligned}$$

$$\theta'_T = \tan^{-1}\frac{T_y}{T_x} = \tan^{-1}\left(\frac{-20.79}{116.98}\right) \approx -10°$$ To the nearest degree

$$\theta_T = \theta'_T \text{ since } T_x > 0,\ |T| = \sqrt{T_x^2 + T_y^2} = \sqrt{116.98^2 + (-20.79)^2} \approx 119$$

Thus, the ground speed of the aircraft is 119 knots and the true course is 10° south of east. ■

The zero vector and the opposite of a vector

It is useful to define a zero vector and the opposite of a vector. The zero vector is defined so its length is zero. The direction does not matter. The opposite of a vector is defined to have equal length but opposite direction. We do this by adding or subtracting 180°. Both definitions are in terms of polar form.

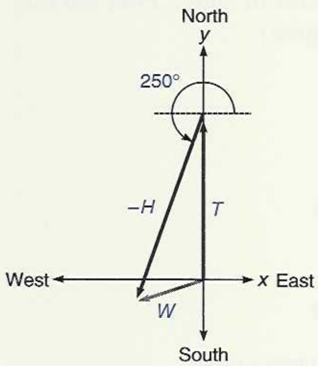
> **Zero vector**
> The vector $0 = (0,\theta)$, where θ is any angle, is the zero vector.
>
> **Opposite of a vector**
> Given a vector $V = (|V|,\theta_V)$, then $-V$ means its opposite, and $-V = (|V|,\ \theta_V \pm 180°)$.

In computing the opposite we generally choose whichever value, $\theta_V + 180°$ or $\theta_V - 180°$, has the smallest absolute value. It is easy to show that for any vector V,

$$V + (-V) = 0$$

Example 8–3 E illustrates uses of the zero vector and the opposite of a vector.

■ Example 8–3 E

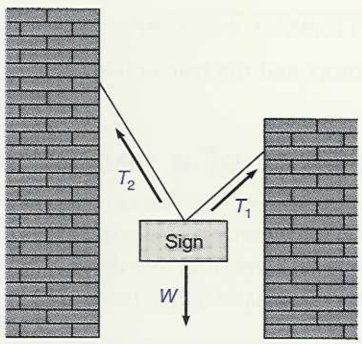


1. An aircraft's on-board inertial navigation computer shows that the aircraft is traveling due north at 200 knots with respect to the ground, and that the aircraft is headed 20° east of north with an airspeed of 225 knots. Find the wind vector W. Interpret this vector.

   We know that true course and ground speed T are due north and 200 knots. Thus, $T = (200,90°)$. Heading and airspeed are 20° east of north and 225 knots. Thus, $H = (225,70°)$.

   $H + W = T$ Aircraft heading + wind = true course
   $W = T - H$ Solve[1] for W
   $= T + (-H)$
   $W = (200,90°) + (225,250°)$ $-H = (225,70° + 180°)$

   Performing the addition shows that $W \approx (77.8,-171.6°)$. This tells us that the wind is blowing in a direction 8.4° south of west at 77.8 knots.



2. A large sign is suspended between two buildings by two wires, as in the figure. One wire acts at an angle of 45° above the horizontal and has a tension (force) T_1, of 400 pounds. If the sign weighs 800 pounds (vector W), compute a vector that describes T_2, the tension and direction of the second wire, to the nearest unit.

   We use a fact from physics to describe the situation. Since the sign is motionless, all the forces acting on it must be balanced, or add to zero. Thus we proceed as follows.

   $T_1 + T_2 + W = 0$ All forces balanced
   $T_2 = -T_1 - W$ Solve for T_2
   $T_1 = (400,45°)$, so $-T_1 = (400,45° + 180°) = (400,225°)$
   $W = (800,270°)$, so $-W = (800,270° - 180°) = (800,90°)$

   Computing $-T_1 - W$ shows that to the nearest unit $T_2 \approx (589,119°)$. ■

Mastery points

Can you

- Find the horizontal and vertical components of a vector?
- Find the magnitude and direction of a vector when given the horizontal and vertical components?
- Add or subtract vectors?
- Apply vectors to navigation and force problems?

[1]We are assuming we can solve a vector-valued equation as we solve real-valued equations. In fact, it can be proved that this is valid.

Exercise 8–3

Convert each vector from its polar to its rectangular form. Leave all answers to the nearest tenth unless the reference angle is 30°, 45°, or 60°. In that case find the exact values.

1. (40,30°) **2.** (15.2,33.6°) **3.** (100.0,122.3°) **4.** (4.2,97.3°)
5. (10.0,200.0°) **6.** (18,120°) **7.** (25,300°) **8.** (82.0,341.9°)
9. (6,−45°) **10.** (5.9,59.2°) **11.** (7.8,−264.3°) **12.** (20.0,−333.0°)

Convert each vector to polar form. Round to the nearest tenth unless an exact form is possible (if the reference angle is 30°, 45°, or 60°).

13. (3.0,4.0) **14.** (31.2,6.9) **15.** (−3.0,5.2) **16.** (−12.5,31.0)
17. $(\sqrt{3},-2)$ **18.** (3,−3) **19.** (5,−10) **20.** (−12.5,−20.3)
21. (−6.8,3.4) **22.** (−8,4) **23.** $(-\sqrt{2},-\sqrt{8})$ **24.** (−1,0.4)

Add the following vectors. Leave the resultant in rectangular form. Round the resultant to the nearest tenth.

25. (−3,8), (2,12) **26.** $(4,3\frac{1}{2})$, $(-1,-2\frac{1}{2})$ **27.** $(\sqrt{2},5)$, $(\sqrt{8},1)$, $(\sqrt{50},-6)$
28. (5,−13), $(-\frac{1}{2},8)$, $(7\frac{1}{2},-1)$

Add the following vectors. Leave the resultant in polar form. Round the resultant to the nearest tenth.

29. (30.0,30°), (15.2,33.6°) **30.** (100,0°), (4.2,97.3°) **31.** (10.0,200°), (29.3,250°)
32. (13.7,300.0°), (82.0,341.9°) **33.** (3.2,−45.0°), (5.9,−59.2°) **34.** (37.9,−100.5°), (69.2,−170.1°)
35. (41.9,−213.0°), (7.7,−264.3°) **36.** (20.0,−333.0°), (39.2,−359.0°) **37.** (3.5,19.2°), (2.7,83.1°), (4.3,145.7°)
38. (7.1,13.8°), (6.2,131°), (10.4,215°) **39.** (15.3,311°), (20.9,117°), (13.2,83°) **40.** (3.2,19.5°), (5.1,45.0°), (6.0,180°)
41. (3.5,−25°), (6.8,25°), (4.2,50°) **42.** (25,−30°), (25,−60°), (25,−100°)

43. An aircraft is moving in the direction 25° north of west at 150 knots. Find the east–west and north–south components of its velocity to the nearest knot and interpret the results.

44. An aircraft is moving in the direction 35° east of south at 120 knots. Find the east–west and north–south components of its velocity to the nearest knot and interpret the results.

45. An aircraft is moving in the direction 15° north of east at 200 knots. Find the east–west and north–south components of its velocity to the nearest knot and interpret the results.

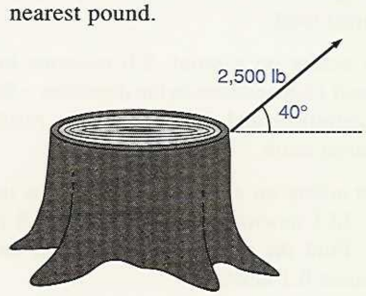
46. A rocket is climbing with a speed of 825 knots and an angle of climb of 58.6°. (The angle of climb is the angle measured from the ground to its flight path.) Find the horizontal and vertical components of the rocket's velocity, to the nearest knot.

47. An aircraft is traveling in a direction 30° west of north. Its speed is 456 knots. Find the east–west and north–south components of its velocity, to the nearest knot.

48. At the location of a ship, the Gulf Stream ocean current is moving to the northwest at a speed of 8.2 knots. Find the east–west and north–south components of its velocity, to the nearest knot.

49. A ship has left an east-coast harbor and has been sailing in a direction 32° north of east for 2.5 hours, at a speed of 18 knots. (a) How far north of the harbor has it gone, to the nearest nautical mile? (b) How far east of the harbor has it gone, to the nearest nautical mile?

50. A force is acting on a tree stump at a 40° angle of elevation (40° measured from the horizontal up to the force vector). See the diagram. If the force is 2,500 pounds, find its vertical and horizontal components, to the nearest pound.



51. A 2,250-pound force is pulling on a sled loaded with lumber at an angle of elevation of 33°. If the sled will not move until the horizontal component of the force exceeds 1,900 pounds, will the sled move?

52. A sled loaded with lumber will not move until the horizontal component of the applied force is 1,200 pounds or more. If a winch being used to move the sled can apply a maximum force of 1,700 pounds, what is the largest angle of elevation at which the winch can act on the sled and move it?

53. Consider a force of 1,000 pounds acting at an angle of elevation of 15° on a point.
- **a.** Compute the horizontal and vertical components of the force.
- **b.** Double the force to 2,000 pounds and recompute the horizontal and vertical components. Do they double also?
- **c.** Double the angle of elevation to 30° (keep the force at 1,000 pounds). Recompute the horizontal and vertical components. Do they double also?

54. A plane is flying over Minneapolis with a ground speed of 200 miles per hour and true course due east. After one hour it turns to a true course of 60° south of east, maintaining the same ground speed. After flying for an additional half-hour the navigator notes its position on a map. How far and in what direction is the plane from Minneapolis, to the nearest unit?

55. A plane is flying over Orlando with ground speed 135 miles per hour, and true course 23° north of east. After 1 hour, it turns to a true course of 40° south of west, maintaining the same ground speed. After flying for an additional hour the navigator notes its position on a map. How far and in what direction is the plane from Orlando, to the nearest unit?

56. Two forces are acting on a point, 12.6 pounds in the direction 123° and 15.8 pounds in the direction 211°. Compute the magnitude and direction of the resultant force to the nearest tenth.

57. Two forces are acting on a point, 2.6 newtons in the direction 18.3° and 15.8 newtons in the direction −86.2°. Compute the magnitude and direction of the resultant force, to the nearest tenth.

58. Three forces are acting on a point: 27.6 newtons in the direction 18.3°, 32.1 newtons at 223.0°, and 46.8 newtons at −30.0°. Find the resultant force acting on the point, to the nearest 0.1 newton.

59. Three forces are acting on a point: 199 pounds at 19.0°, 175 pounds at 131.0°, and 96 pounds at 130.0°. Find the resultant force acting on the point, to the nearest 0.1 pound.

60. A ship leaves its harbor traveling 10° north of east. After 1 hour it turns to the direction 40° south of east. After 2 more hours, it turns to the direction 15° west of south. The ship travels for 1 half-hour more and then stops. The ship has maintained a steady speed of 16 knots (nautical miles per hour) for the entire trip. How many nautical miles, and in what direction, is the ship from its starting position, to the nearest unit?

61. A ship leaves its harbor traveling 15° west of south. After 1 hour it turns to the direction 34° south of west. After 2 more hours, it turns to the direction 10° north of west. The ship travels for 1 half-hour more and then stops. The ship has maintained a steady speed of 20 knots for the entire trip. How many nautical miles, and in what direction, is the ship from its starting position, to the nearest unit?

62. An aircraft is flying with an airspeed of 123 knots and a heading of 30° west of north. The wind is blowing in the direction 15° south of west at 26 knots. Add the heading and wind vectors to find the aircraft's true course and ground speed, to the nearest integer.

63. If the wind in problem 62 now shifts to 35 knots in the direction 10° west of south, find the aircraft's new true course and ground speed, to the nearest integer.

64. A ship is traveling through an ocean current that flows in the direction 5° west of north at 7.2 knots. The ship's heading is 10° north of west, and its speed relative to the water is 19.6 knots. Add these two vectors to find the ship's true course and speed, to the nearest tenth.

65. A ship is traveling through an ocean current that flows in the direction 15° west of north at 7.2 knots. The ship's heading is 10° north of east, and its speed relative to the water is 26.1 knots. Add these two vectors to find the ship's true course and speed, to the nearest tenth.

66. The voltage in an alternating current circuit adds vectorially. If one voltage E_1 is 122 volts with phase angle 30° and a second voltage E_2 is 86 volts with phase angle 21°, find the magnitude and phase angle of the resultant voltage E_T to the nearest unit.

67. (Refer to problem 66.) In an AC circuit E_1 is 240 volts at −45° and E_2 is 115 volts at +45°. Find the resultant E_T to the nearest unit.

68. An aircraft has a ground speed of 135 knots and true course 35° east of north. If its heading is due north and airspeed is 120 knots, find the direction and speed of the wind, to the nearest unit.

69. An aircraft has a ground speed of 80 knots and true course 15° west of north. If the wind is directly from the northeast at 12 knots, find the plane's heading and airspeed, to the nearest unit.

70. A ship is traveling at 14 knots, relative to the water, with heading 20° south of west. If the true speed and direction of the ship is 12 knots due west, find the speed and direction of the ocean current, to the nearest tenth.

71. The ocean current in a certain area is 6.4 knots with direction 8° east of south. A ship in the area is traveling with true direction of 12 knots at 25° east of north. Find the ship's heading and speed relative to the water, to the nearest tenth.

72. Two cables support a one-ton (2,000 pound) sign between two buildings. One of the cables has a tension of 1,500 pounds and acts at an angle of 33° above the horizontal. The other cable is attached to the other building. Find the tension in the other cable as well as its direction relative to the horizontal, to the nearest unit.

73. Two cables support a sign between two buildings. The tension and direction of one cable is 456 pounds at 63° above the horizontal. If the sign weighs 650 pounds, find the tension in the other cable, as well as its direction relative to the horizontal, to the nearest unit.

74. Prove that vector addition is commutative. That is, if A and B are vectors, then $A + B = B + A$. (*Hint:* Real number addition is commutative, and the horizontal and vertical components of a vector are real values.)

75. Prove that vector addition is associative. That is, if A, B, and C are vectors, then $(A + B) + C = A + (B + C)$. (*Hint:* Real number addition is associative, and the horizontal and vertical components of a vector are real values.)

76. Write a program for a computer or graphing calculator that will add two or more vectors when given in polar form.

Skill and review

1. Solve the right triangle ABC if $c = 12.0$ and $A = 23°$.
2. Solve the triangle ABC if $c = 12.0$, $a = 7.5$, and $A = 23°$.
3. Solve the triangle ABC if $c = 12.0$, $a = 7.5$, and $B = 23°$.
4. Graph the rational function $f(x) = \dfrac{5}{x^2 - 4x + 3}$.

8–4 Complex numbers in polar form

$I = \sqrt{\dfrac{P}{Z}}$ is a formula from electronics. Use it to determine I given the complex values $P = 5 + 2i$ and $Z = 1 - 4i$.

Many applications, such as the one above, which require complex numbers are best done with complex numbers in polar form. This is the subject of this section.

Basic definitions

In section 1–7 we made the following definitions.

Imaginary unit

$$i = \sqrt{-1}$$

Complex number (rectangular form)

A number of the form $a + bi$, a and b both real numbers.

Complex conjugate

The complex conjugate of $a + bi$ is $a - bi$.

Equality of complex numbers

$a + bi = c + di$ if and only if $a = c$ and $b = d$.

We also defined addition, subtraction, multiplication, and division for complex numbers.

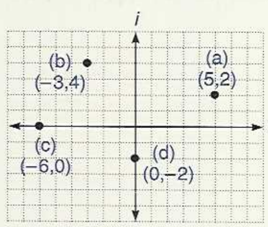


Figure 8–9

Observe that we now refer to the form $a + bi$ as the rectangular form of a complex number. Complex numbers in rectangular form can be graphed as ordered pairs in a rectangular coordinate system by letting horizontal distances represent the real part (a) and vertical distances represent the imaginary part (b) of each complex number. We use a vertical axis marked i and a horizontal axis marked x for this purpose. Figure 8–9 shows the graph of the values (a) $5 + 2i$, (b) $-3 + 4i$, (c) -6, and (d) $-2i$. A graph of complex numbers such as the figure is called an **Argand diagram.**

Polar form of a complex number

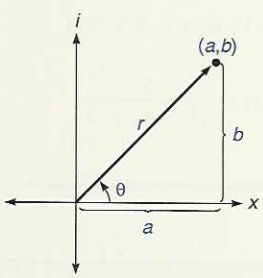


Figure 8–10

We can use the graph of a complex number as a guide to develop another way to represent a complex number. Given the complex number $a + bi$, let r represent the distance from the origin to the point (a,b), and let θ represent the angle in standard position determined by the ray containing the origin and (a,b). See figure 8–10. Using the definitions of section 5–2 we obtain $\cos\theta = \frac{a}{r}$ and $\sin\theta = \frac{b}{r}$. Thus, $a = r\cos\theta$ and $b = r\sin\theta$. This means we can rewrite $a + bi$ as $r\cos\theta + (r\sin\theta)i$, or $r(\cos\theta + i\sin\theta)$. The expression $\cos\theta + i\sin\theta$ occurs so often it is abbreviated as **cis** θ, so $a + bi = r\text{ cis }\theta$. This form is called the **polar form** of a complex number.[2] Also, note that $r = \sqrt{a^2 + b^2}$ and that $\tan\theta = \frac{b}{a}$.

[2]In 1893 Irving Stringham first used the notation $\text{cis }\beta = \cos\beta + i\sin\beta$.

> **Polar form of a complex number**
> If $z = a + bi$ is a complex number that determines an angle θ, then
> $$z = r \text{ cis } \theta$$
> is its polar form, where cis θ means $\cos\theta + i\sin\theta$, and
> $$r = \sqrt{a^2 + b^2}$$
> The value r is called the **modulus** of z, which is also written $|z|$.

Note The value of θ is not unique. All coterminal values produce the same rectangular form. Thus, 2 cis 10° is equivalent to 2 cis 370°.

Polar-rectangular conversions

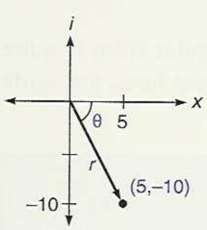
As with vectors (section 8–3), we generally give a value of θ so that $-180° < \theta \le 180°$. A method for converting from rectangular form to polar form is a paraphrase of the method from converting vectors from rectangular to polar form.

> Given a complex number $z = a + bi = r \text{ cis } \theta$,
> $$r = \sqrt{a^2 + b^2}, \quad \tan\theta = \tan^{-1}\frac{b}{a} \text{ (if } a \neq 0\text{)}, \quad \text{and}$$
> $$\theta = \begin{cases} \theta' & \text{if } a > 0 \\ \theta' - 180° \text{ if } \theta' > 0 & \text{if } a < 0 \\ \theta' + 180° \text{ if } \theta' < 0 & \end{cases}$$

Note If $a = 0$ then θ is 90° if $b > 0$, and $-90°$ if $b < 0$. A sketch will make the choice clear.

■ ***Example 8–4 A***

Convert between polar and rectangular form.

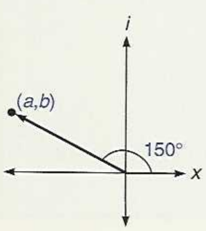


1. $5 - 10i$

The graph is shown in the figure.

$$r = |5 - 10i| = \sqrt{5^2 + (-10)^2} = \sqrt{125} = 5\sqrt{5}$$
$$\theta' = \tan^{-1}\frac{b}{a} = \tan^{-1}\left(\frac{-10}{5}\right) = \tan^{-1}(-2) \approx -63.4°$$
$$\theta = \theta' \approx -63.4° \text{ since } a > 0$$

Thus, the polar form of $5 - 10i$ is $5\sqrt{5}\text{ cis}(-63.4°) \approx 11.2 \text{ cis}(-63.4°)$.



2. $5 \text{ cis } 150° = 5(\cos 150° + i \sin 150°)$

$$= 5\left(-\frac{\sqrt{3}}{2} + i \cdot \frac{1}{2}\right) = -\frac{5\sqrt{3}}{2} + \frac{5}{2}i$$

Thus, $5 \text{ cis } 150° = -\frac{5\sqrt{3}}{2} + \frac{5}{2}i$ or about $-4.3 + 2.5i$. ■

Polar-rectangular conversions on a calculator

As stated in section 8–3 most calculators are programmed to perform polar/rectangular conversions. This conversion works equally well for vectors (section 8–3) and for complex numbers. Typical keystrokes are illustrated here. (The TI-81 steps are shown below.)

Example 8–4 A, part 1 would be done as follows:

$z = 5 - 10i$

5 [R → P] 10 [+/−] [=] Display: [11.18033989]

[$x \leftrightarrow y$] Display: [−63.43494882]

Thus, $z \approx 11.2 \text{ cis}(-63.4°)$.

Example 8–4 A, part 2 would be done in the following way:

$z = 5 \text{ cis } 150°$

5 [P → R] 150 [=] Display: [−4.330127019]

[$x \leftrightarrow y$] Display: [2.5]

Thus, $z \approx -4.3 + 2.5i$.

TI-81 (degree mode)

Example 8–4 A, part 1:

[MATH] 1 5 [ALPHA] [,] [(−)] 10 [)] [ENTER] Display: [11.18033989]

[ALPHA] 3 [ENTER] Display: [−63.43494882]

Example 8–4 A, part 2:

[MATH] 2 5 [ALPHA] [,] 150 [)] [ENTER] Display: [−4.330127019]

[ALPHA] 1 [ENTER] Display: [2.5]

Multiplication and division of complex numbers in polar form

Multiplication and division of complex numbers in rectangular form is quite complicated. The following theorems show that these procedures are quite simple when the complex numbers are in polar form.

Complex multiplication—polar form

$$(r_1 \text{cis } \theta_1)(r_2 \text{cis } \theta_2) = r_1 r_2 \text{cis}(\theta_1 + \theta_2).$$

Complex division—polar form

$$\frac{r_1 \text{cis } \theta_1}{r_2 \text{cis } \theta_2} = \frac{r_1}{r_2} \text{cis}(\theta_1 - \theta_2),\ r_2 \neq 0.$$

Concept

To multiply, multiply the moduli and add the angles. To divide, divide the moduli and subtract the angles.

We can see that the first theorem is true by converting the complex numbers into rectangular form and performing the multiplication as defined in section 1–7.

$(r_1 \text{ cis } \theta_1)(r_2 \text{ cis } \theta_2)$
$= (r_1\cos\theta_1 + ir_1\sin\theta_1)(r_2\cos\theta_2 + ir_2\sin\theta_2)$
$= r_1r_2\cos\theta_1\cos\theta_2 + ir_1r_2\cos\theta_1\sin\theta_2 + ir_1r_2\sin\theta_1\cos\theta_2 + i^2r_1r_2\sin\theta_1\sin\theta_2$
$= r_1r_2\cos\theta_1\cos\theta_2 - r_1r_2\sin\theta_1\sin\theta_2 + ir_1r_2\cos\theta_1\sin\theta_2 + ir_1r_2\sin\theta_1\cos\theta_2$

Recall that $i^2 = -1$

$= r_1r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)]$

Now use the identities for $\cos(\alpha + \beta)$ and $\sin(\alpha + \beta)$ from section 7–2

$= r_1r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$
$= r_1r_2\text{cis}(\theta_1 + \theta_2)$

The proof of the process for division in polar form is left as an exercise.

In electronics, Ohm's law states $V = IZ$, where V means voltage, I means current, and Z means impedance. The units are volts, amperes, and ohms, respectively. Often complex numbers are used to describe the values of volts, amperes, and ohms. The fact that the angles add in forming the product models the physical situation in an electronics circuit. This use is illustrated in example 8–4 B.

■ ***Example 8–4 B***

Multiply or divide the complex numbers.

1. In a certain electronic circuit current $I = 4 \text{ cis } 300°$ amperes and impedance $Z = 2 \text{ cis } 150°$. Use Ohm's law, $V = IZ$, to compute voltage V.

$V = IZ = (4 \text{ cis } 300°)(2 \text{ cis } 150°) = 8 \text{ cis } 450° = 8 \text{ cis } 90°$ (volts)

90° coterminal to 450°

2. $\dfrac{15 \text{ cis } 30°}{18 \text{ cis } 80°} = \dfrac{15}{18}\text{ cis } (30° - 80°) = \dfrac{5}{6}\text{ cis } (-50°)$ ■

De Moivre's theorem

Consider the following computations for successive powers of a complex number r cis θ.

$(r \text{ cis } \theta)^1 = r \text{ cis } \theta$
$(r \text{ cis } \theta)^2 = (r \text{ cis } \theta)(r \text{ cis } \theta) = r^2\text{cis } 2\theta$
$(r \text{ cis } \theta)^3 = (r \text{ cis } \theta)(r \text{ cis } \theta)^2 = (r \text{ cis } \theta)(r^2 \text{ cis } 2\theta) = r^3\text{cis } 3\theta$
$(r \text{ cis } \theta)^4 = (r \text{ cis } \theta)(r \text{ cis } \theta)^3 = (r \text{ cis } \theta)(r^3 \text{ cis } 3\theta) = r^4\text{cis } 4\theta$

It is logical to assume that this pattern continues. This is true, and the result is called **De Moivre's theorem.** It can be proved for positive integers using the method of finite induction in section 12–4. It actually turns out that the exponent can be any real number.

De Moivre's theorem

$(r \text{ cis } \theta)^n = r^n\text{cis } n\theta$ for any real number n.

This theorem is illustrated in example 8–4 C.

■ *Example 8–4 C*

Use De Moivre's theorem to compute $(5 \text{ cis } 137°)^3$; leave the answer in polar form.

$$\begin{aligned} &= 5^3 \text{ cis } (3 \cdot 137°) \\ &= 125 \text{ cis } 411° \\ &= 125 \text{ cis } 51° \end{aligned}$$

De Moivre's theorem for roots

In the complex number system every number except 0 has n nth roots; that is, two square roots, three cube roots, four fourth roots, etc. These can be expressed by De Moivre's theorem by replacing n by $\frac{1}{n}$; recall that $x^{\frac{1}{2}} = \sqrt{x}$, $x^{\frac{1}{3}} = \sqrt[3]{x}$, $x^{\frac{1}{4}} = \sqrt[4]{x}$, etc.

> **De Moivre's theorem for roots**
>
> The n nth roots of $r \text{ cis } \theta$ are of the form
>
> $$r^{\frac{1}{n}} \text{cis}\left(\frac{\theta}{n} + \frac{k \cdot 360°}{n}\right), \; 0 \le k < n$$
>
> where k and n are positive integers.

We can show that any number of the form above is an nth root of $r \text{ cis } \theta$ by raising it to the nth power.

$$\begin{aligned} \left[r^{\frac{1}{n}} \text{cis}\left(\frac{\theta}{n} + \frac{k \cdot 360°}{n}\right)\right] &= \left(r^{\frac{1}{n}}\right)^n \text{cis}\left[n\left(\frac{\theta}{n} + \frac{k \cdot 360°}{n}\right)\right] \\ &= r \text{ cis}(\theta + k \cdot 360°) \\ &= r \text{ cis } \theta \qquad \theta + k \cdot 360° \text{ is coterminal to } \theta \end{aligned}$$

If $k \ge n$, we get a repetition of a previous root. The proof of this is left as an exercise. The proof of the fact that the roots are all distinct and that there are no other roots is beyond the scope of this text.

■ *Example 8–4 D*

Find the roots.

1. Find the three cube roots of 1.

$$1 = 1 \text{ cis } 0°$$

Evaluate $1^{\frac{1}{3}} \text{cis}\left(\frac{0°}{3} + \frac{k \cdot 360°}{3}\right) = \text{cis}(k \cdot 120°)$ for $k = 0, 1, 2$.

$k = 0$: $\text{cis } 0° = \cos 0° + i \sin 0° = 1 + 0i = 1$

$k = 1$: $\text{cis } 120° = \cos 120° + i \sin 120° = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$

$k = 2$: $\text{cis } 240° = \cos 240° + i \sin 240° = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$

Thus the three cube roots of 1 are 1, $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$, and $-\frac{1}{2} - \frac{\sqrt{3}}{2}i$.

2. Find decimal approximations (to tenths) of the four fourth roots of $10 - 2\sqrt{39}i$.

$$10 - 2\sqrt{39}i \approx 16 \text{ cis } 308.68°$$

$$16^{\frac{1}{4}}\text{cis}\left(\frac{308.68°}{4} + \frac{k \cdot 360°}{4}\right) \approx 2 \text{ cis}(77.17 + k \cdot 90°)$$

Evaluate $2 \text{ cis}(77.17 + k \cdot 90°)$ for $k = 0, 1, 2, 3$.

$k = 0$: $2 \text{ cis } 77.17° = 2(\cos 77.17° + i \sin 77.17°) = 0.4 + 2.0i$
$k = 1$: $2 \text{ cis } 167.17° = 2(\cos 167.17° + i \sin 167.17°) = -2.0 + 0.4i$
$k = 2$: $2 \text{ cis } 257.17° = 2(\cos 257.17° + i \sin 257.17°) = -0.4 - 2.0i$
$k = 3$: $2 \text{ cis } 347.17° = 2(\cos 347.17° + i \sin 347.17°) = 2.0 - 0.4i$

Thus, the four fourth roots of $10 - 2\sqrt{39}i$ are approximately $0.4 + 2.0i$, $-2.0 + 0.4i$, $-0.4 - 2.0i$, and $2.0 - 0.4i$. ■

Mastery points

Can you

- Multiply and divide complex numbers in polar form?
- State and use De Moivre's theorem for integral powers?
- State and use De Moivre's theorem for roots?

Exercise 8–4

Write the polar form of each complex number. Round the results to the nearest tenth.

1. $5 - 2i$ **2.** $\sqrt{2} + 3i$ **3.** $-1 + 3i$ $\boxed{4.}$ $\sqrt{3} - 2i$
5. $-3 + 4i$ **6.** $13 - 9i$

Write the polar form of each complex number. Leave the result in exact form.

$\boxed{7.}$ $\sqrt{3} + i$ **8.** $1 - \sqrt{3}i$ **9.** $3 + 3i$ **10.** $-1 + \sqrt{3}i$
11. $-1 - i$ **12.** $\sqrt{5} - \sqrt{5}i$ **13.** $5i$ **14.** $-3i$

Write the rectangular form of the following numbers. Round the results to the nearest tenth.

$\boxed{15.}$ $3 \text{ cis } 15°$ **16.** $5 \text{ cis } 20°$ **17.** $4.5 \text{ cis } 35°$ **18.** $10 \text{ cis } 40°$
19. $\sqrt{2} \text{ cis } 315°$ **20.** $200 \text{ cis } 8°$ **21.** $13.6 \text{ cis}(-25°)$ **22.** $12 \text{ cis}(-6°)$

Write the rectangular form of the following numbers. Leave the result in exact form.

23. $\sqrt{3} \text{ cis } 30°$ **24.** $4 \text{ cis } 210°$ **25.** $10 \text{ cis } 300°$ $\boxed{26.}$ $6 \text{ cis } 135°$
27. $\sqrt{10} \text{ cis } 180°$ **28.** $2 \text{ cis } 90°$ **29.** $\sqrt{8} \text{ cis } 315°$ **30.** $5 \text{ cis } 240°$

Multiply or divide the following complex numbers. Leave the result in polar form.

31. $(5 \text{ cis } 30°)(3 \text{ cis } 45°)$ **32.** $(2 \text{ cis } 18°)(4.5 \text{ cis } 100°)$ **33.** $(5.4 \text{ cis } 300°)(2 \text{ cis } 300°)$

34. $(0.5 \text{ cis } 230°)(80 \text{ cis } 200°)$ **35.** $\dfrac{20 \text{ cis } 100°}{5 \text{ cis } 20°}$ **36.** $\dfrac{100 \text{ cis } 45°}{200 \text{ cis } 15°}$

37. $\dfrac{40 \text{ cis } 80°}{18 \text{ cis } 160°}$ **38.** $\dfrac{90 \text{ cis } 300°}{50 \text{ cis } 100°}$

Use De Moivre's theorem to compute the power indicated. Leave the answer in the form in which the problem is stated (polar or rectangular).

39. $(8 \text{ cis } 100°)^3$

40. $(5 \text{ cis } 10°)^4$

41. $(3 \text{ cis } 200°)^3$

42. $(2 \text{ cis } 300°)^5$

43. $(0.5 - 1.2i)^8$ (round to nearest tenth)

44. $(0.8 + 0.6i)^{10}$ (round to nearest tenth)

Solve the following problems.

45. Find the 3 cube roots of 8 in exact form.

46. Find the 4 fourth roots of -1 in exact form.

47. Find the 4 fourth roots of 81 in exact form.

48. Find the 6 sixth roots of -64 in exact form.

49. Find the 3 cube roots of $75 - 100i$ to the nearest tenth.

50. Find the 4 fourth roots of $\sqrt{3} + 3i$ to the nearest tenth.

51. In electronics, one version of Ohm's law says that $I = \dfrac{V}{Z}$, where I is current, V is voltage, and Z is impedance. Find I in a circuit in which V is 125 cis 25° and Z is 50 cis 45°.

52. Find I in a circuit in which $V = 200$ cis 40° and $Z = 4$ cis 50°. See problem 51.

53. Find V in a circuit where $I = 10$ cis 15° and $Z = 5$ cis 30°. See problem 51.

54. Find Z in a circuit where $I = 40$ cis 200° and $V = 10$ cis 125°. See problem 51.

55. In a parallel electronic circuit with two legs, total impedance Z_T is $\dfrac{Z_1 Z_2}{Z_1 + Z_2}$. Find Z_T in a circuit in which $Z_1 = 2 + i$ and $Z_2 = 3 - 5i$. Leave the answer in polar form.

56. Find Z_T in a parallel circuit in which $Z_1 = 12 + 3i$ and $Z_2 = 4 - 2i$. Leave the answer in polar form. See problem 55.

57. Use $I = \sqrt{\dfrac{P}{Z}}$ to determine I if $P = 5 + 2i$ and $Z = 1 - 4i$. Leave the result in rectangular form, to the nearest hundredth. Use the first square root ($k = 0$ in De Moivre's theorem).

58. Use $I = \sqrt{\dfrac{P}{Z}}$ to determine I if $P = -2 + 2i$ and $Z = 2 - i$. Leave the result in rectangular form, to the nearest hundredth. Use the first square root ($k = 0$ in De Moivre's theorem).

59. Is the following an identity: $a \text{ cis}(-\theta) = -a \text{ cis } \theta$?

Multiplication by i can be interpreted as a 90° rotation. If z represents the complex number given in each of the following problems, compute and graph z, iz, i^2z, i^3z.

60. $4 + 2i$

61. $-3 + i$

62. $5i$

63. 6

64. $1 - i$

65. $-1 - i$

66. Let $z_1 = -2 + 2i$, $z_2 = 1 - \sqrt{3}i$. Form the product in two ways: (a) by multiplication in rectangular form and (b) by changing each value into polar form (in exact form) and performing the multiplication in polar form. Then (c) convert the answer to (b) back to rectangular form and verify that the answers to (a) and (b) are the same.

67. A numerically controlled machine is programmed to rotate a laser beam according to mathematical rules. The laser initially points to the point $1 + i$.

a. Find the rectangular form of a complex number z such that the angle of the product $z(1 + i)$ is 30° greater than the angle of $1 + i$, without changing the modulus of $1 + i$.

b. Give the rectangular form of the point to which the laser points after this rotation. Round the answer to two decimal places.

c. Give the rectangular form of the point to which the laser points after eight such rotations, starting at the point $1 + i$. Round the answer to two decimal places.

68. In this section, we stated that $\dfrac{r_1 \text{cis } \theta_1}{r_2 \text{cis } \theta_2} = \dfrac{r_1}{r_2} \text{ cis}(\theta_1 - \theta_2)$. Prove that this is true. Use the proof in the text that $(r_1 \text{cis } \theta_1)(r_2 \text{cis } \theta_2) = r_1 r_2 \text{cis}(\theta_1 + \theta_2)$ as a guide.

69. *Warning:* This problem requires a *great* deal of algebraic manipulation. De Moivre's theorem for roots states that the n nth roots of r cis θ are of the form

$$(r \text{ cis } \theta)^{\frac{1}{n}} = r^{\frac{1}{n}} \text{cis}\left(\frac{\theta}{n} + \frac{k \cdot 360°}{n}\right),\ 0 \le k < n,$$

where k and n are positive integers. Show that if $k \ge n$, then the expression is a repetition of another root. That is, it is the same as the expression for some value of $k < n$. Do this in the following manner.

First, if $k \ge n$, then $\frac{k}{n} = a + \frac{b}{n}$, where $b < n$. $\left(\text{Think of the example } 22 \ge 5, \text{ so } \frac{22}{5} = 4 + \frac{2}{5}.\right)$ This means $k = an + b$, $b < n$.

Next, show that the following steps are true:

$$r^{\frac{1}{n}} \text{cis}\left(\frac{\theta}{n} + \frac{k \cdot 360°}{n}\right)$$

$$= r^{\frac{1}{n}} \text{cis}\left[\frac{\theta}{n} + \frac{(an + b) \cdot 360°}{n}\right] \quad \text{Why?}$$

$$= r^{\frac{1}{n}} \text{cis}\left[\left(\frac{\theta}{n} + \frac{b \cdot 360°}{n}\right) + a \cdot 360°\right].$$

Show the algebra for this step

Finally expand this last expression into terms of sine and cosine (i.e., expand "cis α"), and then apply the identities for the cosine and sine of a sum (section 7–2).

Skill and review

1. Add the vectors (3.8,28°) and (5.1,134°). Leave in polar form, with the magnitude to the nearest 0.1 and the angle to the nearest integer.
2. If vector $A = (190,30°)$ and $C = (150,68°)$, and $A + B = C$, find vector B in polar form. Leave results to the nearest unit.
3. Solve triangle ABC if $a = 12.6$, $b = 19.1$, and $c = 28.0$. Round answers to tenths.
4. Rationalize the denominator of $\frac{1}{\sqrt[3]{4a^4b}}$.
5. If $\tan \theta = 1 + u$ and θ terminates in quadrant I, describe $\sin \theta$ in terms of u.

8–5 Polar coordinates

The pattern of strongest radiation of a certain radio antenna is described by the equation $r = 1 + 2 \sin 2\theta$. This pattern is said to have sidelobes. Convert this equation into rectangular form (rewrite it in terms of x- and y-coordinates).

Some natural phenomena, such as the motion of the planets, the contour of a cam on an automobile camshaft, the path traveled by a client on many of the rides in an amusement park, or the field strength around a radio transmitter, can be described most simply by describing the phenomenon in terms of distance from some point as a line moves in a circle. The polar coordinate system is a coordinate system that is better suited to describing these phenomena than are rectangular coordinates.[3] A polar coordinate system is a series of concentric circles and an angle reference line (see figure 8–11). The common center of the circles is called the **pole.**

[3]James Bernoulli is often credited with the creation of polar coordinates in 1691, although Isaac Newton used them earlier.

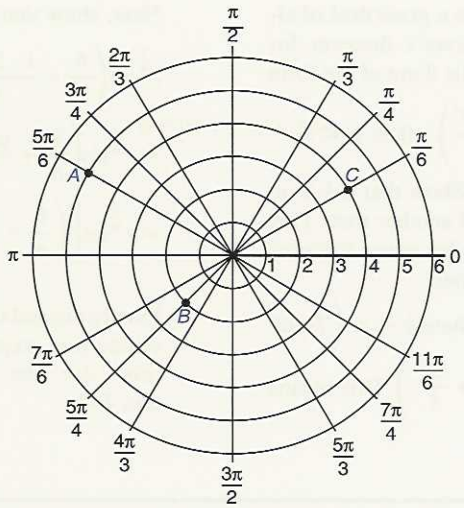


Figure 8–11

Definitions

Polar coordinates

The polar coordinates of a point are an ordered pair of the form (r,θ), where r is the *radius* and θ is an angle.

A point in polar form is located by finding the radius line corresponding to the angle θ, often stated in radians, and moving r units from the center along this line. Figure 8–11 shows the graphs[4] of the points $\left(5,\frac{5\pi}{6}\right)(A)$, $\left(2,\frac{5\pi}{4}\right)(B)$, and $\left(4,\frac{\pi}{6}\right)(C)$. In each of these cases $r > 0$.

If two points have equal radii and coterminal angles, they will have the same graph. For this reason we define such points to be equivalent. If $r < 0$ we interpret this to mean a change of direction by π radians (the opposite direction).

Equivalence of points

1. $(r,\alpha) = (r,\beta)$ if α and β are coterminal angles or $r = 0$.
2. $(-r,\theta) = (r,\theta + \pi)$.

Note $(-r,\theta) = (r,\theta - \pi)$ because $\theta + \pi$ and $\theta - \pi$ are coterminal.

[4]Polar coordinate paper is widely available.

This definition means that $(1,0)$, $(1,2\pi)$, $(1,4\pi)$, $(1,-2\pi)$, $(1,-4\pi)$, $(-1,\pi)$, $(-1,3\pi)$, $(-1,-\pi)$, etc. all describe the same point! This can occasionally lead to some confusion.

Example 8–5 A illustrates the basic definitions.

■ ***Example 8–5 A***

1. List 2 other coordinates for the point $\left(2,\frac{3\pi}{4}\right)$, with one so that $r < 0$.

$$\left(2,\frac{3\pi}{4}\right) = \left(2,\frac{3\pi}{4} + 2\pi\right) = \left(2,\frac{11\pi}{4}\right)$$ Adding 2π gives a coterminal angle

$$\left(2,\frac{3\pi}{4}\right) = \left(-2,\frac{3\pi}{4} + \pi\right) = \left(-2,\frac{7\pi}{4}\right).$$ Adding or subtracting π gives a coterminal angle in which r changes sign

2. Plot the point $\left(-5,\frac{11\pi}{6}\right)$.

$$\left(-5,\frac{11\pi}{6}\right) = \left(5,\frac{11\pi}{6} - \pi\right) = \left(5,\frac{5\pi}{6}\right),$$

which is plotted at A in figure 8–11. ■

Polar-rectangular coordinate conversions

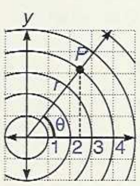


Figure 8–12

There is a way to relate polar and rectangular coordinates. Figure 8–12 shows polar and rectangular coordinates superimposed. From the definitions of section 5–3 we know that if $P = (x,y) = (r,\theta)$, $r > 0$, then $\cos\theta = \frac{x}{r}$ and $\sin\theta = \frac{y}{r}$.

Thus, *to convert from polar to rectangular coordinates* we have only to use

$$x = r\cos\theta$$
$$y = r\sin\theta$$

In fact, it will be an exercise to show that we can use the same relations when $r < 0$. Example 8–5 B illustrates a polar-to-rectangular conversion.

■ ***Example 8–5 B***

Convert the polar coordinates $\left(2,\frac{\pi}{3}\right)$ to rectangular coordinates.

$$x = r\cos\theta = 2\cos\frac{\pi}{3} = 2\cdot\frac{1}{2} = 1$$

$$y = r\sin\theta = 2\sin\frac{\pi}{3} = 2\cdot\frac{\sqrt{3}}{2} = \sqrt{3}$$

Thus, the rectangular coordinates are $(1,\sqrt{3})$. ■

To convert from rectangular to polar coordinates we use the fact that $r^2 = x^2 + y^2$ and $\tan\theta = \frac{y}{x}$ if $x \neq 0$. This also means $\theta' = \tan^{-1}\frac{y}{x}$.

As with vectors and complex numbers (sections 8–3 and 8–4) we always leave the angle θ so that it is the smallest possible absolute value. In this case, using radian measure, we always choose θ so that $-\pi < \theta \leq \pi$.

A rule for finding θ is essentially the same as that for finding θ_V for vectors and θ for complex numbers.

> Given polar coordinates for point P, $P = (x,y) = (r,\theta)$,
>
> $$r = \sqrt{x^2 + y^2}, \quad \tan\theta' = \tan^{-1}\frac{y}{x} \text{ if } x \neq 0, \quad \text{and}$$
>
> $$\theta = \begin{cases} \theta' & \text{if } x > 0 \\ \theta' - \pi \text{ if } \theta' > 0 & \text{if } x < 0 \\ \theta' + \pi \text{ if } \theta' < 0 & \end{cases}$$

Note If $x = 0$ then θ is π if $y > 0$, and $-\pi$ if $y < 0$. This is clear from a sketch.

■ ***Example 8–5 C***

Convert the rectangular coordinates into polar coordinates.

1. $(-2\sqrt{3},2)$

$r^2 = (-2\sqrt{3})^2 + 2^2 = 16$, $r = 4$

$$\theta' = \tan^{-1}\frac{2}{-2\sqrt{3}} = \tan^{-1}\left(-\frac{\sqrt{3}}{3}\right) = -\tan^{-1}\frac{\sqrt{3}}{3}$$

$\tan^{-1}$ is an odd function, so $\tan^{-1}(-x) = -\tan^{-1} x$

so $\theta' = -\dfrac{\pi}{6}$. $x < 0$, $\theta' < 0$, so $\theta = \theta' + \pi = -\dfrac{\pi}{6} + \pi = \dfrac{5\pi}{6}$.

Therefore, the required polar coordinates are $\left(4,\dfrac{5\pi}{6}\right)$.

2. $(-2,-5)$

$$r^2 = (-2)^2 + (-5)^2 = 29,\ r = \sqrt{29} \approx 5.39$$
$$\theta' = \tan^{-1}\tfrac{5}{2} \approx 1.19 \text{ (radians)}$$

$x < 0$, $\theta' > 0$, so $\theta = \theta' - \pi = \tan^{-1}\frac{5}{2} - \pi$ (exactly), or approximately $1.19 - \pi \approx -1.95$.

Thus, the polar coordinates are $(\sqrt{29},\tan^{-1}2.5 - \pi)$ exactly, or approximately $(5.39,-1.95)$. ■

Conversions with a calculator

Engineering/scientific calculators are programmed to perform the conversions of example 8–5 C, using the same method and keys shown in section 8–3 for vectors and in section 8–4 for complex numbers. These calculators have keys marked R → P and P → R, or something equivalent. The only difference is

that the calculator must be in radian mode. See the sections for keystokes for polar to rectangular conversions (example 8–5 B) and rectangular to polar conversions (example 8–5 C).

Conversion of equations between rectangular and polar form

Recall that a nonvertical line in analytic geometry is an equation of the form $y = mx + b$, and a circle with center at the origin and radius r is an equation of the form $x^2 + y^2 = r^2$. These and other equations can also be written using polar coordinates using the variables r and θ.

It is often useful to discover the polar coordinate version of a rectangular coordinate equation. We say that *a rectangular equation and a polar equation are equivalent if they describe the same set of points,* assuming the appropriate rectangular/polar conversions of the points themselves.

Conversion of equations from rectangular to polar form

To convert an equation in rectangular form into an equivalent equation in polar form, use the relations used to convert a point from rectangular to polar form:

$$x = r\cos\theta, \quad y = r\sin\theta, \quad \text{and } r^2 = x^2 + y^2$$

When possible we customarily write a polar equation in which r is described as a function of θ. That is, to the extent possible, we put all terms with r in one member of the equation, and all other terms in the other member.

Example 8–5 D illustrates conversions from rectangular to polar form.

■ ***Example 8–5 D***

Convert each rectangular equation into polar form.

1. The line $y = 3x - 2$.

$$y = 3x - 2$$

$$r\sin\theta = 3(r\cos\theta) - 2 \qquad x = r\cos\theta,\ y = r\sin\theta$$

$$2 = 3r\cos\theta - r\sin\theta$$

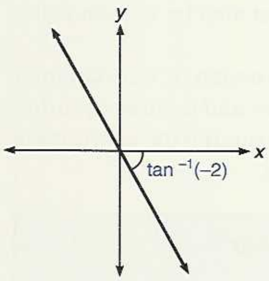
It is customary to solve a polar equation for r if possible.

$$2 = r(3\cos\theta - \sin\theta)$$

$$r = \frac{2}{3\cos\theta - \sin\theta}$$

2. The line $y = -2x$.

$$y = -2x$$
$$r \sin \theta = -2r \cos \theta \qquad x = r\cos\theta,\ y = r\sin\theta$$
$$\sin \theta = -2 \cos \theta \qquad \text{Divide both members by } r\text{; this assumes } r \neq 0 \text{ (see below)}$$
$$\frac{\sin \theta}{\cos \theta} = -2 \qquad \text{Divide both members by } \cos\theta\text{; this assumes } \cos\theta \neq 0 \text{ (see below)}$$
$$\tan \theta = -2$$



This equation is equivalent to the equation $y = -2x$. To see this, first observe that it does not mention r. This means r can be any value. The values of θ for which $\tan \theta = -2$ are in quadrants II and IV, where the tangent function takes on negative values. All points (r,θ) for which $\tan \theta = -2$ and r takes on any value are shown in the figure. This is the line $y = -2x$.

It was all right to assume $r \neq 0$ here because the resulting solution includes the pole as a solution (this is where $r = 0$).

It was also valid to assume $\cos \theta \neq 0$, for two reasons. One is that we arrive at an equation that satisfies the requirements and thus we do not need to consider the case where $\cos \theta = 0$. Second, when $\cos \theta = 0$, $\sin \theta$ is ± 1. These values do not solve the equation $\sin \theta = -2 \cos \theta$, so that $\cos \theta = 0$ cannot occur in this situation.

3. The hyperbola $x^2 - y^2 = 2$. (Hyperbolas are discussed in section 11–3.)

$$x^2 - y^2 = 2$$
$$(r \cos \theta)^2 - (r \sin \theta)^2 = 2$$
$$r^2\cos^2\theta - r^2\sin^2\theta = 2$$
$$r^2(\cos^2\theta - \sin^2\theta) = 2$$
$$r^2\cos 2\theta = 2 \qquad \cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$r^2 = \frac{2}{\cos 2\theta}$$
$$r^2 = 2 \sec 2\theta$$

Conversion of equations from polar to rectangular form

To convert from polar to rectangular coordinates we use the relations

$$\sin \theta = \frac{y}{r}, \quad \cos \theta = \frac{x}{r}, \quad r^2 = x^2 + y^2$$

Example 8–5 E illustrates converting polar equations into rectangular form.

■ ***Example 8–5 E***

Convert the polar equation into rectangular form.

1. $r = 5 \sec \theta$

$$r = \frac{5}{\cos \theta}$$
$$r \cos \theta = 5$$
$$r \cdot \frac{x}{r} = 5$$
$$x = 5$$

2. $r^2 = \cos 2\theta$

We do not have any relation for cos 2θ, so we use an identity (section 7–3) to replace it.

$$r^2 = \cos^2\theta - \sin^2\theta$$
$$r^2 = \left(\frac{x}{r}\right)^2 - \left(\frac{y}{r}\right)^2$$
$$r^2 = \frac{x^2}{r^2} - \frac{y^2}{r^2}$$
$$r^4 = x^2 - y^2$$
$$(x^2 + y^2)^2 = x^2 - y^2$$

3. $r = 1 + \cos \theta$

$$r = 1 + \frac{x}{r}$$
$$r^2 = r + x$$
$$x^2 + y^2 = r + x$$
$$x^2 + y^2 - x = r$$
$$(x^2 + y^2 - x)^2 = r^2$$
$$(x^2 + y^2 - x)^2 = x^2 + y^2$$
$$x^4 - 2x^3 + y^4 + 2x^2y^2 - 2xy^2 - y^2 = 0$$

Note Observe that r can be replaced by squaring both members as necessary to obtain r^2, or any even power of r.

Mastery points

Can you

- Graph points in the polar coordinate system?
- Give alternate polar coordinates for a given point?
- Convert between polar and rectangular coordinates?
- Convert equations between polar and rectangular form?

Exercise 8–5

Graph the following points in polar coordinates.

1. $(3,0)$ **2.** $(4,\pi)$ **3.** $\left(2,\frac{\pi}{6}\right)$ **4.** $\left(1,\frac{\pi}{3}\right)$ **5.** $\left(2,\frac{3\pi}{4}\right)$ **6.** $\left(3,\frac{7\pi}{6}\right)$

7. $\left(6,\frac{11\pi}{6}\right)$ **8.** $\left(5,\frac{\pi}{2}\right)$ **9.** $(-2,\pi)$ **10.** $\left(-4,\frac{5\pi}{3}\right)$ 11. $\left(-1,\frac{\pi}{3}\right)$ **12.** $\left(-5,\frac{\pi}{6}\right)$

13. $(4,2)$ **14.** $(3,5)$ $\boxed{15.}$ $(-5,6)$ **16.** $\left(-4,\frac{\pi}{2}\right)$ **17.** $\left(5,\frac{4\pi}{3}\right)$ **18.** $\left(4,\frac{5\pi}{6}\right)$

List three of the many other coordinates for the point, with one point having $r < 0$ and two points having $r > 0$.

19. $\left(2,\frac{\pi}{6}\right)$ **20.** $\left(1,\frac{\pi}{3}\right)$ **21.** $\left(6,\frac{11\pi}{6}\right)$ **22.** $\left(-5,\frac{\pi}{6}\right)$ **23.** $(2,2)$ **24.** $\left(4,\frac{17\pi}{6}\right)$

Convert the following polar coordinates into rectangular coordinates. Leave the result in exact form.

25. $\left(4,\frac{\pi}{2}\right)$ **26.** $\left(2,\frac{\pi}{3}\right)$ **27.** $\left(5,\frac{5\pi}{6}\right)$ **28.** $\left(1,\frac{11\pi}{6}\right)$ **29.** $\left(4,\frac{4\pi}{3}\right)$ **30.** $\left(2,\frac{5\pi}{3}\right)$

Convert the following polar coordinates into rectangular coordinates to two-decimal place accuracy.

31. $(2,1)$ **32.** $(5,1.2)$ **33.** $(3,0.82)$ **34.** $(3,5)$ **35.** $(4,4)$ **36.** $(1,6)$

Convert the following rectangular coordinates into polar coordinates. Leave the result in exact form.

37. $(-2\sqrt{3},-2)$ **38.** $(3,-3)$ **39.** $(-2,0)$ **40.** $(-1,\sqrt{3})$ **41.** $(-4,-4)$ **42.** $(0,1)$

Convert the following rectangular coordinates into polar coordinates to two-decimal place accuracy.

43. $(2,3)$ **44.** $(-5,2)$ **45.** $(1,-4)$ **46.** $(-4,-3)$ **47.** $(5,4)$ **48.** $(-3,5)$

Convert the following rectangular equations into polar equations.

49. $y = 4x$ **50.** $y = -2x$ **51.** $y = -3x + 2$ **52.** $y = 5x - 3$ **53.** $y = mx + b, b \neq 0$
54. $y = 2$ **55.** $y^2 - 2x^2 = 5$ **56.** $y^2 - x = 4$ **57.** $3x^2 + 2y^2 = 1$ **58.** $x^2 + y^2 = 3$

Convert the following polar equations into rectangular equations.

59. $r = \sin\theta$ **60.** $2r = \cos\theta$ **61.** $r = 2\sec\theta$ **62.** $r = 3\csc\theta$
63. $r = 3\sin 2\theta$ **64.** $r = 2\cos\theta$ **65.** $r^2 = \sin 2\theta$ **66.** $r = \cos 2\theta$
67. $r^2 = \tan\theta$ **68.** $r\sin\theta = 5$ **69.** $r = \dfrac{3}{1 - 2\sin\theta}$ **70.** $r = \dfrac{5}{4 - \cos\theta}$

71. Show that a polar equation of $2xy = 5$ is $r^2 = 5\csc 2\theta$.

72. In the text we noted that $x = r\cos\theta$ if $r > 0$. Show that this is also true for a point given in polar coordinates where $r < 0$. (*Hint:* Consider a point $P = (r,\theta)$, where $r < 0$. Then $P = (-r,\theta + \pi)$, where $-r > 0$. Therefore, since $-r > 0$, $x = -r\cos(\theta + \pi)$ is true. Proceed from here.)

73. In the text we noted that $y = r\sin\theta$ if $r > 0$. Show that this is also true for a point given in polar coordinates where $r < 0$. See the hint in problem 72.

74. The shape of a cam that drives a certain sewing machine needle is described by the polar equation $r = 3 - 2\cos\theta$. Convert this equation into rectangular form.

75. The path that an industrial robot must follow to paint a pattern on a part being manufactured is described by the curve $r = 1 - 2\sin\theta$. Convert this equation into rectangular form.

76. The pattern of strongest radiation of a certain bidirectional radio antenna is described by the curve $r = 1 + \sin 2\theta$. Convert this equation into rectangular form.

77. The pattern of strongest radiation of a certain radio antenna is described by the equation $r = 1 + 2\sin 2\theta$. This pattern is said to have sidelobes. Convert this equation into rectangular form.

78. In the October 1983 issue of *Scientific American,* Jearl Walker described several rides, the Scrambler and the Calypso, at the Geauga Lake Amusement Park near Cleveland, Ohio. Assume the path taken by the Scrambler is described by the polar equation $r = 2\cos 3\theta$. Convert this equation into rectangular form. It will be necessary to rewrite $\cos 3\theta$ in terms of $\cos\theta$. See problem 82 in section 7–3.

79. Assume the path of the Calypso (problem 78) is described by the equation $r = 1 - 3\cos\theta$. Convert this equation into rectangular form.

Skill and review

1. Multiply the complex numbers 2 cis 30° and 5 cis 45°.
2. Find the exact form of the 3 cube roots of 1,000.
3. Solve the triangle ABC if $a = 125$, $b = 85$, and $C = 50°$. Leave all results to the nearest integer.
4. Put the function $f(x) = 2x^2 + 4x - 5$ into vertex form (by completing the square) and graph.
5. Combine $\dfrac{2x-3}{x^2-16} - \dfrac{5}{x-4}$.
6. Divide $\dfrac{2+3i}{5-2i}$.
7. Simplify $\left(\dfrac{x^2x^{-5}}{x^3}\right)^{-2}$.
8. Multiply $3x^{-2}(\frac{1}{3}x^2 - 2x + 1)$. Rewrite the result so there are no negative exponents, and combine into one term.

Chapter 8 summary

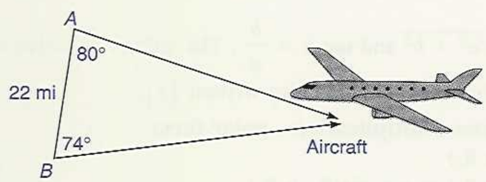
- **The law of sines** In any triangle ABC, $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$. This assumes side a is opposite angle A, side b is opposite angle B, side c is opposite angle C.
- **The law of cosines** For any triangle ABC,
  $a^2 = b^2 + c^2 - 2bc \cos A$
  $b^2 = a^2 + c^2 - 2ac \cos B$
  $c^2 = a^2 + b^2 - 2ab \cos C$
  This assumes side a is opposite angle A, side b is opposite angle B, and side c is opposite angle C.
- **Polar form of a vector** A vector A in polar form is the ordered pair $A = (|A|, \theta_A)$.
  $|A|$ is the magnitude of vector A; $|A| \geq 0$.
  θ_A is the direction of vector A.
- **Rectangular form of a vector** A vector A in rectangular form is $A = (A_x, A_y)$.
  A_x is the horizontal component of the vector,
  A_y is the vertical component of the vector.
- **Relation between the polar and rectangular form of a vector** Given vector $A = (|A|, \theta_A) = (A_x, A_y)$,
  $A_x = |A| \cos \theta_A$
  $A_y = |A| \sin \theta_A$
- **Vector sum** Let Z be the resultant (vector sum) of two vectors A and B. Then $Z = A + B$, and if
  $A = (A_x, A_y)$ and $B = (B_x, B_y)$, then
  $Z_x = A_x + B_x$ and $Z_y = A_y + B_y$
- **Polar form of a complex number** If $z = a + bi$ is a complex number that determines an angle θ, the $r \text{ cis } \theta$ is its polar form, where cis θ means $\cos \theta + i \sin \theta$, $r = \sqrt{a^2 + b^2}$ and $\tan \theta = \dfrac{b}{a}$. The value r is called the modulus of z, which is also written $|z|$.
- **Complex multiplication—polar form**
  $(r_1 \text{ cis } \theta_1)(r_2 \text{ cis } \theta_2) = r_1 r_2 \text{ cis}(\theta_1 + \theta_2)$
- **Complex division—polar form**
  $\dfrac{r_1 \text{ cis } \theta_1}{r_2 \text{ cis } \theta_2} = \dfrac{r_1}{r_2} \text{cis}(\theta_1 - \theta_2)$, $r_2 \neq 0$
- **De Moivre's theorem** $(r \text{ cis } \theta)^n = r^n \text{cis } n\theta$ for any real number n.
- **De Moivre's theorem for roots** The n nth roots of r cis θ are of the form $(r \text{ cis } \theta)^{\frac{1}{n}} = r^{\frac{1}{n}} \text{cis}\left(\dfrac{\theta}{n} + \dfrac{k \cdot 360°}{n}\right)$, $0 \leq k < n$, where k and n are positive integers.
- The polar coordinates of a point are an ordered pair of the form (r, θ), where r is the radius, or radius vector, and θ is an angle.
- **Equivalence of points in polar coordinates**
  1. $(r, \alpha) = (r, \beta)$, if α and β are coterminal angles.
  2. $(-r, \theta) = (r, \theta + \pi)$.
- **Relation between polar and rectangular coordinates** If $P = (x,y) = (r,\theta)$, $r > 0$, then $x = r \cos \theta$ and $y = r \sin \theta$.
- **To convert a rectangular equation into a polar equation** use the relations $x = r \cos \theta$, $y = r \sin \theta$, and $r^2 = x^2 + y^2$.
- **To convert a polar equation into a rectangular equation** use the relations $\sin \theta = \dfrac{y}{r}$, $\cos \theta = \dfrac{x}{r}$, and $r^2 = x^2 + y^2$.

Chapter 8 review

[8–1] Solve the following oblique triangles using the law of sines. Round answers to the nearest tenth.

1. $a = 10.6$, $A = 47.9°$, $B = 10.3°$
2. $b = 3.55$, $B = 23.8°$, $C = 5.2°$
3. $a = 10.0$, $b = 13.0$, $B = 79.0°$
4. $a = 12.6$, $c = 7.0$, $C = 32.7°$

5. A lost pilot is given a position report by triangulation with two radar sites. The situation is shown in the diagram. Find the distance of the aircraft from radar site A, to the nearest mile.



[8–2] Solve the following oblique triangles to the nearest tenth. Use the law of cosines.

6. $a = 4.1$, $b = 6.8$, $C = 29.4°$
7. $b = 60.0$, $c = 20.0$, $A = 92.1°$
8. $a = 21.4$, $c = 27.0$, $B = 112°$
9. $a = 43.5$, $b = 17.8$, $c = 35.0$
10. $a = 31.7$, $b = 80.0$, $c = 105$

11. A technician is setting up a numerically controlled grinding machine. A triangular pattern is to be ground and, therefore, must be coordinatized. The vertices of the triangle are at $A(-2,5)$, $B(4,7)$, and $C(5,2)$. Solve the resulting triangle. Round answers to the nearest 0.1.

12. A ship leaves a harbor heading due west and travels 23.3 km. It then turns north through a 63° angle and travels for another 10.0 km. How far is the ship from its starting point, to the nearest kilometer?

[8–3]

13. Given the vector (27.2,29.0°), find the horizontal and vertical components, to the nearest tenth.

14. The horizontal and vertical components of a vector are 19.6 and 30.5, respectively. Find the magnitude and direction of the vector.

15. A rocket climbs with a speed of 450 knots and an angle of climb of 34.6°. (The angle of climb is the angle measured from the ground to its flight path.) Find the horizontal and vertical components of the rocket's velocity, to the nearest knot.

16. A force is acting on a cart at a 13.5° angle of elevation (13.5° measured from the ground up to the force vector). If the force is 256 pounds, find the horizontal and vertical components, to the nearest pound.

Add the following vectors. Round results to the nearest tenth.

17. (33.0,14.7°), (15.2,33.6°)
18. (10.2,112.3°), (4.2,19.3°)
19. (3.5,29.2°), (1.7,43.1°), (4.3,115.0°)
20. (7.1,13.8°), (6.6,142.0°), (11.9,215.0°)

21. Two forces are acting on a point, 126 pounds in the direction 223° and 158 pounds in the direction 311°. Compute the magnitude and direction of the resultant force, to the nearest unit.

[8–4] Write the polar form of each complex number. Round answers to the nearest tenth when necessary.

22. $3 - 2i$ **23.** $\sqrt{3} + 3i$ **24.** $-1 - 2i$

Write the rectangular form of each complex number. Round answers to the nearest tenth.

25. 3 cis 35° **26.** 5 cis 243°

Write the rectangular form of each complex number. Leave results in exact form.

27. 3 cis 240° **28.** 10 cis 330°

Multiply the complex numbers.

29. (2 cis 25°)(3 cis 45°) **30.** (2 cis 18°)(6.5 cis 122°)

Divide the complex numbers.

31. $\dfrac{40 \text{ cis } 120°}{5 \text{ cis } 20°}$ **32.** $\dfrac{50 \text{ cis } 45°}{100 \text{ cis } 9°}$

33. Compute the cube of 2 cis 130°.

34. Compute the fourth power of 2 cis 150°.

35. Compute an approximation to $(0.8 + 0.6i)^8$; round results to the nearest 0.01.

36. Find the 4 fourth roots of 16 in exact rectangular form.

37. Find the 3 cube roots of $6 - 5i$ to the nearest 0.01. Leave the answer in rectangular form.

38. In electronics, one version of Ohm's law is $I = \dfrac{V}{Z}$, where I is current, V is voltage, and Z is impedance. Find I in a circuit in which $V = 130$ cis 25° and $Z = 30$ cis 75°.

[8–5] Graph the following points in polar coordinates.

39. (2,0) **40.** $(2,\pi)$ **41.** $\left(3,\dfrac{5\pi}{6}\right)$ **42.** $\left(-6,\dfrac{11\pi}{6}\right)$

Convert the following polar coordinates into rectangular coordinates. Leave the result in exact form.

43. $\left(3,\frac{11\pi}{6}\right)$ **44.** $\left(4,\frac{2\pi}{3}\right)$ **45.** $\left(-2,\frac{5\pi}{3}\right)$

Convert the following polar coordinates into rectangular coordinates. Round results to the nearest tenth.

46. $\left(2,\frac{\pi}{5}\right)$ **47.** $(5,2)$ **48.** $(-1,-4)$

Convert the following rectangular coordinates into polar coordinates. Round results to the nearest hundredth.

49. $(2,1)$ **50.** $(-5,3)$ **51.** $(-1,-4)$

Convert the following rectangular equations into polar equations.

52. $y = -3x$ **53.** $y = 4x + 2$ **54.** $2y^2 - x^2 = 5$
55. $y^2 - 3x = 0$ **56.** $x^2 + y^2 = 9$ **57.** $x = 2$

Convert the following polar equations into rectangular equations.

58. $r = \sin\theta$ **59.** $r = 2\sec\theta$ **60.** $r^2 = \sin 2\theta$
61. $r^2 = \tan\theta$ **62.** $r\sin\theta = 2$ **63.** $r = \dfrac{3}{2 - \sin\theta}$

Chapter 8 test

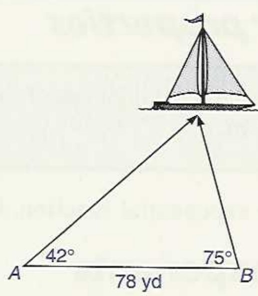
1. In triangle ABC, $b = 22.6$, $A = 13.5°$, and $C = 82.1°$. Solve this triangle. Round answers to the nearest tenth.

2. In triangle ABC, $b = 22.6$, $c = 24.0$, and $C = 62.1°$. Solve this triangle. Round answers to the nearest tenth.

3. In triangle ABC, $a = 25.9$, $c = 16.2$, and $B = 100°$. Solve this triangle. Round answers to the nearest tenth.

4. In triangle ABC, $a = 2.55$, $b = 3.12$, and $c = 4.00$. Solve this triangle. Round answers to the nearest tenth.

5. The distance to a boat on a lake is being found by triangulation from two points on shore. The situation is shown in the diagram. Find the distance to the boat from site B to the nearest yard.



6. Given the three points $A(6,8)$, $B(-3,5)$, and $C(10,-4)$, find the angle formed by line segments AB and BC, to the nearest 0.1°.

7. Find the horizontal and vertical components of the vector $(2,30°)$, in exact form.

8. The horizontal and vertical components of a vector are 4.0 and 5.0. Find the magnitude and direction of the vector, to the nearest tenth.

9. Add the vectors $(5.4,19.0°)$ and $(8.0,123°)$. Round the results to the nearest tenth.

10. A sign is suspended between two buildings. One cable from which the sign is suspended has a tension of 535 pounds and acts at an angle of elevation of 62° (i.e., 62° with respect to the horizontal). If the weight of the sign is 1,000 pounds, find the tension and angle of elevation in the other cable, to the nearest unit.

11. Write the polar form of the complex number $4 - 5i$. Round to hundredths.

12. Write the rectangular form of the complex number 2 cis 120°. Leave the result in exact form.

13. Multiply (2 cis 325°)(7 cis 145°). Leave the result in polar form.

14. Divide $\dfrac{6 \text{ cis } 140°}{2 \text{ cis } 20°}$. Leave the result in polar form.

15. Compute $(3 \text{ cis } 150°)^3$.

16. Find the 4 fourth roots of -16 in exact form.

17. Graph the polar coordinates $\left(-4,\frac{\pi}{6}\right)$.

18. Convert the polar coordinates $\left(3,\frac{\pi}{8}\right)$ to rectangular coordinates, to the nearest tenth.

19. Convert the rectangular coordinates $(-\sqrt{3},-1)$ into polar coordinates, in exact form.

20. Convert the rectangular equation $y = -3x + 5$ into polar form.

21. Convert the rectangular equation $2y^2 - x = 5$ into polar form.

22. Convert the polar equation $r = 2\csc\theta$ into rectangular form.

23. Convert the polar equation $r^2 = \cos 2\theta$ into rectangular form.

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